

Exercise 3.3. Calculate the eigenvectors and eigenvalues of  $\sigma_n$ . *Hint:* Assume the eigenvector  $\lambda_1$  has the form

$$\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$$

where  $\alpha$  is an unknown parameter. Plug this vector into the eigenvalue equation and solve for  $\alpha$  in terms of  $\theta$ . Why did we use a single parameter  $\alpha$ ? Notice that our suggested column vector must have unit length.

Matrix  $\sigma_n$  is given on page 85.

$$\sigma_n = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

Equation (3.5) is the eigenvalue equation where  $\lambda$  is an eigenvalue,  $|\lambda\rangle$  is an eigenvector, and  $\mathbf{M}$  is a matrix.

$$\mathbf{M}|\lambda\rangle = \lambda|\lambda\rangle \tag{3.5}$$

Following the above hint we have

$$\sigma_n|\lambda_1\rangle = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \alpha + \sin \theta \sin \alpha \\ \sin \theta \cos \alpha - \cos \theta \sin \alpha \end{pmatrix}$$

By sine and cosine angle difference formulas

$$\sigma_n|\lambda_1\rangle = \begin{pmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{pmatrix}$$

Then by equation (3.5) we have

$$\begin{pmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{pmatrix} = \lambda_1 \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$$

Taking  $\lambda_1 = 1$  we have

$$\theta - \alpha = \alpha$$

Hence

$$\alpha = \frac{\theta}{2}$$