2.1. Solve the Klein-Gordon equation.

This is the Klein-Gordon equation.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+\frac{m^2c^2}{\hbar^2}\right)\psi=0$$

One solution is

$$\psi=\exp\left(-\frac{i}{\hbar}(Et-p_x x-p_y y-p_z z)\right) \tag{1}$$

where

$$E=\sqrt{p_x^2c^2+p_y^2c^2+p_z^2c^2+m^2c^4}$$

Let us inspect dimensions. The dimensions of Et are joule-seconds.

$$Et\propto \text{joule}\times\text{second}$$

Hence $Et/\hbar$ is dimensionless.

$$\frac{Et}{\hbar}\propto\frac{\text{joule}\,\text{second}}{\text{joule}\,\text{second}}=1$$

The dimensions of $p_x x$ are joule-seconds.

$$p_x x\propto\frac{\text{kilogram}\,\text{meter}}{\text{second}}\times\text{meter}=\text{joule}\,\text{second}$$

Hence $p_x x/\hbar$ is dimensionless.

$$\frac{p_x x}{\hbar}\propto\frac{\text{joule}\,\text{second}}{\text{joule}\,\text{second}}=1$$

Solution (1) can be written in a more compact form as

$$\psi=\exp\left(-\frac{i}{\hbar}P\cdot X\right)$$

where

$$P=\begin{pmatrix}E/c\\p_x\\p_y\\p_z\end{pmatrix},\quad X=\begin{pmatrix}ct\\x\\y\\z\end{pmatrix}$$

and

$$P\cdot X=P^\mu g_{\mu\nu}X^\nu=Et-p_x x-p_y y-p_z z$$