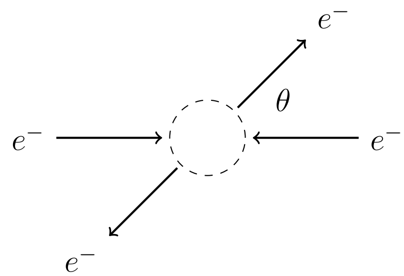
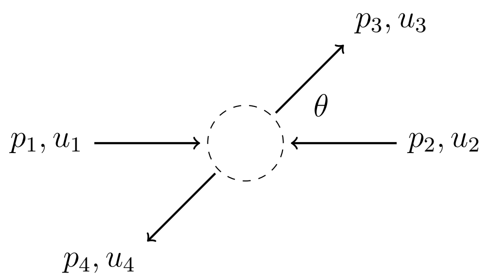
The following diagram shows the geometry of a typical collider experiment that generates electron scattering data.



Here is the same diagram with momentum labels p and Dirac spinor labels u.



In center of mass coordinates the momentum vectors are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix} \qquad p_3 = \begin{pmatrix} E \\ p\sin\theta\cos\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -p\sin\theta\cos\phi \\ -p\sin\theta\sin\phi \\ -p\cos\theta \end{pmatrix}$$

particle 1   particle 2   particle 3   particle 4

Symbol p is incident momentum, E is total energy $E = \sqrt{p^2 + m^2}$, and m is electron mass. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

The spinors are

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix} \qquad u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix} \qquad u_{21} = \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix} \qquad u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}$$

particle 1 spin up   particle 1 spin down   particle 2 spin up   particle 2 spin down

$$u_{31} = \begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix} \qquad u_{32} = \begin{pmatrix} 0 \\ E+m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix} \qquad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix} \qquad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

particle 3 spin up   particle 3 spin down   particle 4 spin up   particle 4 spin down

The spinors shown above are not individually normalized. Instead, a combined spinor normalization constant $N=(E+m)^4$ will be used.

The following formula computes a probability density $|\mathcal{M}_{abcd}|^2$ for electron scattering where the subscripts $abcd$ are the spin states of the electrons.

$$|\mathcal{M}_{abcd}|^2 = \frac{e^4}{N}\left|\underbrace{\frac{1}{t}(\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b})}_{\substack{\text{from Feynman diagram with}\\ \text{photon exchange}\\ \text{no electron interchange}}} - \underbrace{\frac{1}{u}(\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})}_{\substack{\text{from Feynman diagram with}\\ \text{photon exchange}\\ \text{electron interchange}}}\right|^2$$

Symbol e is electron charge. Symbols t and u are Mandelstam variables

$$t=(p_1-p_3)^2$$
$$u=(p_1-p_4)^2$$

Let

$$a_1=(\bar{u}_{3c}\gamma^\mu u_{1a})(\bar{u}_{4d}\gamma_\mu u_{2b}) \qquad a_2=(\bar{u}_{4d}\gamma^\nu u_{1a})(\bar{u}_{3c}\gamma_\nu u_{2b})$$

Then

$$\begin{aligned}
|\mathcal{M}_{abcd}|^2 &= \frac{e^4}{N}\left|\frac{a_1}{t}-\frac{a_2}{u}\right|^2\\
&= \frac{e^4}{N}\left(\frac{a_1}{t}-\frac{a_2}{u}\right)\left(\frac{a_1}{t}-\frac{a_2}{u}\right)^*\\
&= \frac{e^4}{N}\left(\frac{a_1a_1^*}{t^2}-\frac{a_1a_2^*}{tu}-\frac{a_1^*a_2}{tu}+\frac{a_2a_2^*}{u^2}\right)
\end{aligned}$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}
\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2\\
&= \frac{e^4}{4N}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left(\frac{a_1a_1^*}{t^2}-\frac{a_1a_2^*}{tu}-\frac{a_1^*a_2}{tu}+\frac{a_2a_2^*}{u^2}\right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$\begin{aligned}
f_{11} &= \frac{1}{N}\sum_{abcd}a_1a_1^* = \operatorname{Tr}\left((\not p_3+m)\gamma^\mu(\not p_1+m)\gamma^\nu\right)\operatorname{Tr}\left((\not p_4+m)\gamma_\mu(\not p_2+m)\gamma_\nu\right)\\
f_{12} &= \frac{1}{N}\sum_{abcd}a_1a_2^* = \operatorname{Tr}\left((\not p_3+m)\gamma^\mu(\not p_1+m)\gamma^\nu(\not p_4+m)\gamma_\mu(\not p_2+m)\gamma_\nu\right)\\
f_{22} &= \frac{1}{N}\sum_{abcd}a_2a_2^* = \operatorname{Tr}\left((\not p_4+m)\gamma^\mu(\not p_1+m)\gamma^\nu\right)\operatorname{Tr}\left((\not p_3+m)\gamma_\mu(\not p_2+m)\gamma_\nu\right)
\end{aligned}$$

Hence

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4}\left(\frac{f_{11}}{t^2}-\frac{f_{12}}{tu}-\frac{f_{12}^*}{tu}+\frac{f_{22}}{u^2}\right)$$

The following formulas are equivalent to the Casimir trick. (Recall that $a \cdot b = a^\mu g_{\mu\nu} b^\nu$)

$$\begin{aligned}
f_{11} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_4)^2 - 64m^2(p_1 \cdot p_3) + 64m^4 \\
f_{12} &= -32(p_1 \cdot p_2)^2 + 32m^2(p_1 \cdot p_2) + 32m^2(p_1 \cdot p_3) + 32m^2(p_1 \cdot p_4) - 32m^4 \\
f_{22} &= 32(p_1 \cdot p_2)^2 + 32(p_1 \cdot p_3)^2 - 64m^2(p_1 \cdot p_4) + 64m^4
\end{aligned}$$

In Mandelstam variables

$$\begin{aligned}
s &= (p_1 + p_2)^2 \\
t &= (p_1 - p_3)^2 \\
u &= (p_1 - p_4)^2
\end{aligned}$$

the formulas are

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 - 64sm^2 - 64um^2 + 192m^4 \\
f_{12} &= -8s^2 + 64sm^2 - 96m^4 \\
f_{22} &= 8s^2 + 8t^2 - 64sm^2 - 64tm^2 + 192m^4
\end{aligned}$$

High energy approximation

When $E \gg m$ a useful approximation is to set $m = 0$ and obtain

$$\begin{aligned}
f_{11} &= 8s^2 + 8u^2 \\
f_{12} &= -8s^2 \\
f_{22} &= 8s^2 + 8t^2
\end{aligned}$$

Hence

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{f_{12}}{tu} - \frac{f_{12}^*}{tu} + \frac{f_{22}}{u^2} \right) \\
&= \frac{e^4}{4} \left(\frac{8s^2 + 8u^2}{t^2} - \frac{-8s^2}{tu} - \frac{-8s^2}{tu} + \frac{8s^2 + 8t^2}{u^2} \right) \\
&= 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right)
\end{aligned}$$

Combine terms so $\langle |\mathcal{M}|^2 \rangle$ has a common denominator.

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{u^2(s^2 + u^2) + 2s^2 tu + t^2(s^2 + t^2)}{t^2 u^2} \right)$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned}
s &= 4E^2 \\
t &= 2E^2(\cos\theta - 1) \\
u &= -2E^2(\cos\theta + 1)
\end{aligned}$$

Hence

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{32E^8\cos^4\theta+192E^8\cos^2\theta+288E^8}{16E^8(\cos\theta-1)^2(\cos\theta+1)^2}\right)\\ &= 4e^4\frac{\left(\cos^2\theta+3\right)^2}{(\cos\theta-1)^2(\cos\theta+1)^2}\\ &= 4e^4\left(\frac{\cos^2\theta+3}{\cos^2\theta-1}\right)^2\end{aligned}$$

The following equivalent formula can also be used.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= 2e^4\left(\frac{s^2+u^2}{t^2}+\frac{2s^2}{tu}+\frac{s^2+t^2}{u^2}\right)\\ &= 2e^4\Bigg(\underbrace{\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}}_{\substack{\text{from Feynman diagram with}\\ \text{photon exchange}\\ \text{no electron interchange}}}+\underbrace{\frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)}}_{\text{interaction term}}+\underbrace{\frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}}_{\substack{\text{from Feynman diagram with}\\ \text{photon exchange}\\ \text{electron interchange}}}\Bigg)\end{aligned}$$

For example, see A. Zee, p. 134.

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{\langle|\mathcal{M}|^2\rangle}{64\pi^2 s}=\frac{e^4}{16\pi^2 s}\left(\frac{\cos^2\theta+3}{\cos^2\theta-1}\right)^2,\quad s\gg m$$

Substituting $e^4=16\pi^2\alpha^2$ yields

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{s}\left(\frac{\cos^2\theta+3}{\cos^2\theta-1}\right)^2$$

We can integrate $d\sigma$ to obtain a cumulative distribution function. Recall that

$$d\Omega=\sin\theta\,d\theta\,d\phi$$

Hence

$$d\sigma=\frac{\alpha^2}{s}\left(\frac{\cos^2\theta+3}{\cos^2\theta-1}\right)^2\sin\theta\,d\theta\,d\phi$$

Let $I(\theta)$ be the following integral of $d\sigma$.

$$\begin{aligned}I(\theta) &= \frac{s}{2\pi\alpha^2}\int_0^{2\pi}\int d\sigma\\ &= \int\left(\frac{\cos^2\theta+3}{\cos^2\theta-1}\right)^2\sin\theta\,d\theta,\quad a\le\theta\le\pi-a\end{aligned}$$

Angular support is limited to an arbitrary $a>0$ because $I(0)$ and $I(\pi)$ are undefined. Assume that $I(\theta)-I(a)$ is computable given θ by either symbolic or numerical integration.

Let C be the normalization constant

$$C = I(\pi - a) - I(a)$$

Then the cumulative distribution function $F(\theta)$ is

$$F(\theta) = \frac{I(\theta) - I(a)}{C}, \quad a \le \theta \le \pi - a$$

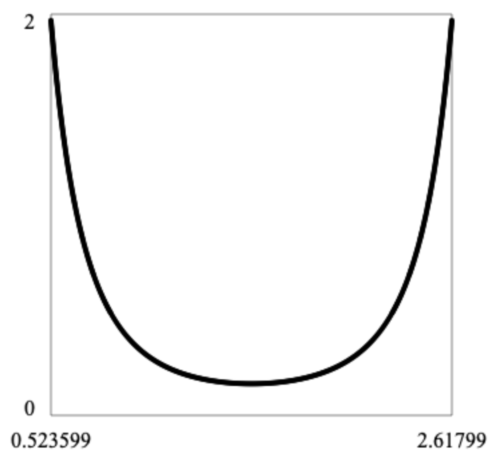
The probability of observing scattering events in the interval θ_1 to θ_2 can now be computed.

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

Probability density function $f(\theta)$ is the derivative of $F(\theta)$.

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{1}{C}\left(\frac{\cos^2\theta + 3}{\cos^2\theta - 1}\right)^2 \sin\theta$$

This is a graph of $f(\theta)$ for $a = \pi/6 = 30°$.



Probability distribution for 30° bins ($a = 30°$).

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	30°	–
30°	60°	0.40
60°	90°	0.10
90°	120°	0.10
120°	150°	0.40
150°	180°	–

Notes

In component notation, the trace operators of the Casimir trick become sums over a repeated index, in this case α.

$$f_{11} = \left((\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)$$

$$f_{12} = (\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$f_{22} = \left((\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha\right)\left((\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha\right)$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$$\gamma^{\beta\mu}{}_\rho \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

$$\gamma^\beta{}_{\mu\rho} \quad \rightarrow \quad \texttt{gammaL = transpose(dot(gmunu,gamma))}$$

Define the following 4×4 matrices.

$$(\not{p}_1 + m) \quad \rightarrow \quad \texttt{X1 = pslash1 + m I}$$

$$(\not{p}_2 + m) \quad \rightarrow \quad \texttt{X2 = pslash2 + m I}$$

$$(\not{p}_3 + m) \quad \rightarrow \quad \texttt{X3 = pslash3 + m I}$$

$$(\not{p}_4 + m) \quad \rightarrow \quad \texttt{X4 = pslash4 + m I}$$

Then for f_{11} we have the following Eigenmath code. The contract function sums over α.

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X3,gammaT,X1,gammaT),1,4)}$$

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X4,gammaL,X2,gammaL),1,4)}$$

Next, multiply then sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$$f_{11} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

Follow suit for f_{22}.

$$(\not{p}_4 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha \quad \rightarrow \quad \texttt{T1 = contract(dot(X4,gammaT,X1,gammaT),1,4)}$$

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not{p}_2 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha \quad \rightarrow \quad \texttt{T2 = contract(dot(X3,gammaL,X2,gammaL),1,4)}$$

Then

$$f_{22} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu)\,\mathrm{Tr}(\cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(dot(T1,transpose(T2)))}$$

The calculation of f_{12} begins with

$$(\not{p}_3 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not{p}_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not{p}_4 + m)^\tau{}_\delta \gamma_\mu{}^\delta{}_\eta (\not{p}_2 + m)^\eta{}_\xi \gamma_\nu{}^\xi{}_\alpha$$

$$\rightarrow \quad \texttt{T = contract(dot(X3,gammaT,X1,gammaT,X4,gammaL,X2,gammaL),1,6)}$$

Then sum over repeated indices μ and ν.

$$f_{12} = \mathrm{Tr}(\cdots \gamma^\mu \cdots \gamma^\nu \cdots \gamma_\mu \cdots \gamma_\nu) \quad \rightarrow \quad \texttt{contract(contract(T,1,3))}$$