15.7.1. Consider a set of modes of the electromagnetic field in which the electric field is polarized along the x direction and the magnetic field is polarized in the y direction. Restricting consideration only to those modes, find the simplest expression you can for the commutation relation $[\hat{E}_x, \hat{B}_y]$ for this multimode field.

Consider the following formulas.

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_\lambda \left(\hat{a}_\lambda - \hat{a}_\lambda^\dagger\right) \sqrt{\frac{\hbar\omega_\lambda}{2\epsilon_0}} \mathbf{u}_\lambda(\mathbf{r}) \tag{15.132}$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \sum_\lambda \left(\hat{a}_\lambda + \hat{a}_\lambda^\dagger\right) \sqrt{\frac{\hbar\omega_\lambda\mu_0}{2}} \mathbf{v}_\lambda(\mathbf{r}) \tag{15.135}$$

By hypothesis we have for the polarization vectors

$$\mathbf{u}_\lambda(\mathbf{r}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_\lambda(\mathbf{r}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence

$$\hat{E}_x(\mathbf{r}, t) = i \sum_\lambda \left(\hat{a}_\lambda - \hat{a}_\lambda^\dagger\right) \sqrt{\frac{\hbar\omega_\lambda}{2\epsilon_0}}$$

$$\hat{B}_y(\mathbf{r}, t) = \sum_\lambda \left(\hat{a}_\lambda + \hat{a}_\lambda^\dagger\right) \sqrt{\frac{\hbar\omega_\lambda\mu_0}{2}}$$

It follows that

$$\hat{E}_x \hat{B}_y = i\frac{\hbar}{2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_\lambda \omega_\lambda \left(\hat{a}_\lambda - \hat{a}_\lambda^\dagger\right)\left(\hat{a}_\lambda + \hat{a}_\lambda^\dagger\right) \tag{1}$$

$$\hat{B}_y \hat{E}_x = i\frac{\hbar}{2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_\lambda \omega_\lambda \left(\hat{a}_\lambda + \hat{a}_\lambda^\dagger\right)\left(\hat{a}_\lambda - \hat{a}_\lambda^\dagger\right) \tag{2}$$

Subtract (2) from (1) to obtain the commutator.

$$[\hat{E}_x, \hat{B}_y] = i\frac{\hbar}{2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_\lambda \omega_\lambda \left(2\hat{a}_\lambda\hat{a}_\lambda^\dagger - 2\hat{a}_\lambda^\dagger\hat{a}_\lambda\right)$$

Recalling that $\hat{a}_\lambda\hat{a}_\lambda^\dagger - \hat{a}_\lambda^\dagger\hat{a}_\lambda = 1$ we have

$$[\hat{E}_x, \hat{B}_y] = i\hbar\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_\lambda \omega_\lambda$$