

15.6.3. Consider the uncertainty relation for position and momentum in the coherent state of Eq. (15.94).

- (i) Deduce the uncertainty relation for the operators $\hat{\xi}_\lambda$ and $\hat{\pi}_\lambda$.
- (ii) By evaluating $\langle \hat{\xi}_\lambda^2 \rangle - \bar{\xi}_\lambda^2$ for the coherent state, show that the standard deviation of the width of the resulting probability distribution for the “position” ξ is $1/\sqrt{2}$, independent of time.
- (iii) Repeat the calculation in (ii) for “momentum”, with operator $\hat{\pi}_\lambda$, instead of “position”.
- (iv) Deduce that the coherent state is a “minimum uncertainty” state, i.e., it has the minimum possible product of the standard deviations of “position” and “momentum”.

[Hints: use the results of Probs. 15.6.1 and 15.6.2, and the general relations for uncertainty principles in Chapter 5.]

(i) We will need the following formulas.

$$[\hat{A}, \hat{B}] = i\hat{C} \quad (5.4)$$

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \frac{\langle \hat{C} \rangle^2}{4} \quad (5.23)$$

From problem 15.5.1 we have

$$[\hat{\xi}_\lambda, \hat{\pi}_\lambda] = i$$

Hence by (5.4) and (5.23) with $\hat{C} = 1$ we have

$$(\Delta\hat{\xi}_\lambda)^2(\Delta\hat{\pi}_\lambda)^2 \geq \frac{1}{4}$$

Take the square root to obtain the uncertainty in terms of standard deviation.

$$\Delta\hat{\xi}_\lambda\Delta\hat{\pi}_\lambda \geq \frac{1}{2}$$

(ii) We have for the coherent state

$$\langle \hat{\xi}_\lambda^2 \rangle = \langle \Psi_{\lambda\bar{n}} | \hat{\xi}_\lambda \hat{\xi}_\lambda | \Psi_{\lambda\bar{n}} \rangle = \frac{1}{2} \langle \Psi_{\lambda\bar{n}} | (\hat{a}_\lambda + \hat{a}_\lambda^\dagger) (\hat{a}_\lambda + \hat{a}_\lambda^\dagger) | \Psi_{\lambda\bar{n}} \rangle$$

From problem 15.6.2 we have

$$\begin{aligned}
\langle \Psi_{\lambda\bar{n}} | \hat{a}_\lambda \hat{a}_\lambda | \Psi_{\lambda\bar{n}} \rangle &= \bar{n} \exp(-2i\omega_\lambda t) \\
\langle \Psi_{\lambda\bar{n}} | \hat{a}_\lambda^\dagger \hat{a}_\lambda^\dagger | \Psi_{\lambda\bar{n}} \rangle &= \bar{n} \exp(2i\omega_\lambda t) \\
\langle \Psi_{\lambda\bar{n}} | \hat{a}_\lambda^\dagger \hat{a}_\lambda | \Psi_{\lambda\bar{n}} \rangle &= \bar{n} \\
\langle \Psi_{\lambda\bar{n}} | \hat{a}_\lambda \hat{a}_\lambda^\dagger | \Psi_{\lambda\bar{n}} \rangle &= \langle \Psi_{\lambda\bar{n}} | (1 + \hat{a}_\lambda^\dagger \hat{a}_\lambda) | \Psi_{\lambda\bar{n}} \rangle = 1 + \bar{n}
\end{aligned}$$

Hence

$$\langle \hat{\xi}_\lambda^2 \rangle = \frac{\bar{n}}{2} \exp(-2i\omega_\lambda t) + \frac{\bar{n}}{2} \exp(2i\omega_\lambda t) + \bar{n} + \frac{1}{2}$$

From problem 15.6.2 we have

$$\langle \hat{\xi}_\lambda \rangle = \sqrt{\frac{\bar{n}}{2}} \exp(-i\omega_\lambda t) + \sqrt{\frac{\bar{n}}{2}} \exp(i\omega_\lambda t)$$

Hence the variance of $\hat{\xi}_\lambda$ is

$$\langle \hat{\xi}_\lambda^2 \rangle - \langle \hat{\xi}_\lambda \rangle^2 = \frac{1}{2}$$

and the standard deviation is

$$\Delta \hat{\xi}_\lambda = \sqrt{\langle \hat{\xi}_\lambda^2 \rangle - \langle \hat{\xi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

(iii) We have for the coherent state

$$\langle \hat{\pi}_\lambda^2 \rangle = \langle \Psi_{\lambda\bar{n}} | \hat{\pi}_\lambda \hat{\pi}_\lambda | \Psi_{\lambda\bar{n}} \rangle = -\frac{1}{2} \langle \Psi_{\lambda\bar{n}} | (\hat{a}_\lambda^\dagger - \hat{a}_\lambda) (\hat{a}_\lambda^\dagger - \hat{a}_\lambda) | \Psi_{\lambda\bar{n}} \rangle$$

From the results in part (ii) we have

$$\langle \hat{\pi}_\lambda^2 \rangle = \bar{n} - \frac{\bar{n}}{2} \exp(-2i\omega_\lambda t) - \frac{\bar{n}}{2} \exp(2i\omega_\lambda t) + \frac{1}{2}$$

For the expectation of $\hat{\pi}_\lambda$ we have

$$\begin{aligned}
\langle \hat{\pi}_\lambda \rangle &= \langle \Psi_{\lambda\bar{n}} | \hat{\pi}_\lambda | \Psi_{\lambda\bar{n}} \rangle \\
&= \frac{i}{\sqrt{2}} \langle \Psi_{\lambda\bar{n}} | (\hat{a}_\lambda^\dagger - \hat{a}_\lambda) | \Psi_{\lambda\bar{n}} \rangle \\
&= i \sqrt{\frac{\bar{n}}{2}} \exp(i\omega_\lambda t) - i \sqrt{\frac{\bar{n}}{2}} \exp(-i\omega_\lambda t)
\end{aligned}$$

It follows that

$$\langle \hat{\pi}_\lambda \rangle^2 = \bar{n} - \frac{n}{2} \exp(2i\omega_\lambda t) - \frac{n}{2} \exp(-2i\omega_\lambda t)$$

Hence the variance of $\hat{\pi}_\lambda$ is

$$\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2 = \frac{1}{2}$$

and the standard deviation is

$$\Delta \hat{\pi}_\lambda = \sqrt{\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

(iv) In part (ii) we found the standard deviation for $\hat{\xi}_\lambda$.

$$\Delta \hat{\xi}_\lambda = \sqrt{\langle \hat{\xi}_\lambda^2 \rangle - \langle \hat{\xi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

In part (iii) we found the standard deviation for $\hat{\pi}_\lambda$.

$$\Delta \hat{\pi}_\lambda = \sqrt{\langle \hat{\pi}_\lambda^2 \rangle - \langle \hat{\pi}_\lambda \rangle^2} = \frac{1}{\sqrt{2}}$$

Hence the uncertainty relation is

$$\Delta \hat{\xi}_\lambda \Delta \hat{\pi}_\lambda = \frac{1}{2}$$

which is the minimum uncertainty from part (i).