15.1.2. Given that

$$\hat{a}^\dagger \hat{a}|\psi_n\rangle = n|\psi_n\rangle$$

and

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$$

show that

$$\hat{a}^\dagger \hat{a}\left(\hat{a}^\dagger|\psi_n\rangle\right) = (n+1)\left(\hat{a}^\dagger|\psi_n\rangle\right)$$

Noting that linear operators are associative we have

$$\hat{a}^\dagger \hat{a}\left(\hat{a}^\dagger|\psi_n\rangle\right) = \hat{a}^\dagger(\hat{a}\hat{a}^\dagger)|\psi_n\rangle$$

Then by the commutator relation given above we have

$$\hat{a}^\dagger \hat{a}\left(\hat{a}^\dagger|\psi_n\rangle\right) = \hat{a}^\dagger(\hat{a}^\dagger \hat{a} + 1)|\psi_n\rangle$$

By the number operator given above we have

$$\hat{a}^\dagger \hat{a}\left(\hat{a}^\dagger|\psi_n\rangle\right) = \hat{a}^\dagger(n+1)|\psi_n\rangle$$

Numbers commute with operators hence

$$\hat{a}^\dagger \hat{a}\left(\hat{a}^\dagger|\psi_n\rangle\right) = (n+1)\hat{a}^\dagger|\psi_n\rangle$$