15.1.1. Prove the relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

for the harmonic oscillator raising and lowering operators, starting from their definitions

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \quad \text{and} \quad \hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right)$$

For $\hat{a}\hat{a}^\dagger$ we have

$$\hat{a}\hat{a}^\dagger\psi = \frac{1}{2}\left(\frac{d}{d\xi} + \xi\right)\left(-\frac{d}{d\xi} + \xi\right)\psi$$

Expand the right-hand side.

$$\begin{aligned}\hat{a}\hat{a}^\dagger\psi &= \frac{1}{2}\left(-\frac{d^2}{d\xi^2}\psi + \frac{d}{d\xi}(\xi\psi) - \xi\frac{d}{d\xi}\psi + \xi^2\psi\right) \\ &= \frac{1}{2}\left(-\frac{d^2}{d\xi^2}\psi + \psi + \xi^2\psi\right) \qquad (1)\end{aligned}$$

For $\hat{a}^\dagger\hat{a}$ we have

$$\hat{a}^\dagger\hat{a}\psi = \frac{1}{2}\left(-\frac{d}{d\xi} + \xi\right)\left(\frac{d}{d\xi} + \xi\right)\psi$$

Expand the right-hand side.

$$\begin{aligned}\hat{a}^\dagger\hat{a}\psi &= \frac{1}{2}\left(-\frac{d^2}{d\xi^2}\psi - \frac{d}{d\xi}(\xi\psi) + \xi\frac{d}{d\xi}\psi + \xi^2\psi\right) \\ &= \frac{1}{2}\left(-\frac{d^2}{d\xi^2}\psi - \psi + \xi^2\psi\right) \qquad (2)\end{aligned}$$

Subtract (2) from (1).

$$\hat{a}\hat{a}^\dagger\psi - \hat{a}^\dagger\hat{a}\psi = \frac{1}{2}\psi - \left(-\frac{1}{2}\psi\right) = \psi$$

Hence

$$[\hat{a}, \hat{a}^\dagger] = 1$$