Surface integral

A surface integral is like adding up all the wind on a sail. In other words, we want to compute

$$\iint \mathbf{F} \cdot \mathbf{n} \, dA$$

where $\mathbf{F} \cdot \mathbf{n}$ is the amount of wind normal to a tiny parallelogram dA. The integral sums over the entire area of the sail. Let S be the surface of the sail parameterized by x and y. (In this model, the z direction points downwind.) By the properties of the cross product we have the following for the unit normal $\mathbf{n}$ and for dA.

$$\mathbf{n} = \frac{\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}}{\left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right|} \qquad dA = \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right| \, dx \, dy$$

Hence

$$\iint \mathbf{F} \cdot \mathbf{n} \, dA = \iint \mathbf{F} \cdot \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right) \, dx \, dy$$

The following exercise is from *Advanced Calculus* by Wilfred Kaplan, p. 313. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

where $\mathbf{F} = xy^2z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \leq 1$ and $\mathbf{n}$ is upper.

Note that the surface intersects the xy plane in a circle. By the right hand rule, crossing x into y yields $\mathbf{n}$ pointing upwards hence

$$\mathbf{n} \, d\sigma = \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right) \, dx \, dy$$

The following code computes the surface integral. The symbols f and h are used as temporary variables.

```
z = 1 - x^2 - y^2
F = (x y^2 z, -2 x^3, y z^2)
S = (x,y,z)
f = dot(F,cross(d(S,x),d(S,y)))
h = sqrt(1 - x^2)
defint(f, y, -h, h, x, -1, 1)
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$\frac{1}{48}\pi$