(8.1) Show that the form of the time evolution operator $\hat{U}(t_2, t_1) = \exp[-i\hat{H}(t_2 - t_1)]$ (as given in eqn 8.7) exhibits properties 1–5 in Section 8.1.

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(1) $\hat{U}(t_1, t_1) = 1$
(2) $\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)$
(3) $i\frac{d}{dt_2}\hat{U}(t_2, t_1) = \hat{H}\hat{U}(t_2, t_1)$
(4) $\hat{U}(t_1, t_2) = \hat{U}^{-1}(t_2, t_1)$
(5) $\hat{U}^{\dagger}(t_2, t_1)\hat{U}(t_2, t_1) = 1$

(1)

$$\begin{aligned}\hat{U}(t_1, t_1) &= \exp[-i\hat{H}(t_1 - t_1)]\\ &= \exp(0)\\ &= 1\end{aligned}$$

(2)

$$\begin{aligned}\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) &= \exp[-i\hat{H}(t_3 - t_2)]\exp[-i\hat{H}(t_2 - t_1)]\\ &= \exp[-i\hat{H}(t_3 - t_1)]\\ &= \hat{U}(t_3, t_1)\end{aligned}$$

(3)

$$\begin{aligned}i\frac{d}{dt_2}\hat{U}(t_2, t_1) &= i\frac{d}{dt_2}\exp[-i\hat{H}(t_2 - t_1)]\\ &= i(-i\hat{H})\exp[-i\hat{H}(t_2 - t_1)]\\ &= \hat{H}\hat{U}(t_2, t_1)\end{aligned}$$

(4)

$$\begin{aligned}\hat{U}(t_1, t_2) &= \exp[-i\hat{H}(t_1 - t_2)]\\ &= \exp[i\hat{H}(t_2 - t_1)]\\ &= \exp[-i\hat{H}(t_2 - t_1)]^{-1}\\ &= \hat{U}^{-1}(t_2, t_1)\end{aligned}$$

(5)

$$
\begin{aligned}
\hat{U}^{\dagger}(t_2, t_1)\hat{U}(t_2, t_1) &= \exp[-i\hat{H}(t_2 - t_1)]^{-1} \exp[-i\hat{H}(t_2 - t_1)] \\
&= \exp[i\hat{H}(t_2 - t_1)] \exp[-i\hat{H}(t_2 - t_1)] \\
&= \exp(0) \\
&= 1
\end{aligned}
$$