(5.3) Show that the commutator of two Hermitian operators $\hat{A}$ and $\hat{B}$ is anti-Hermitian, i.e., that

$$[\hat{A}, \hat{B}]^\dagger = -[\hat{A}, \hat{B}] \tag{5.61}$$

The factor of i in many commutator expressions (e.g. $[\hat{x}, \hat{p}] = i\hbar$, $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, and $[\hat{A}, \hat{B}] = \frac{1}{i\hbar}\{A, B\}_{\text{PB}}$) makes sure that this property is obeyed.

We have

$$\begin{aligned}
[\hat{A}, \hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger \\
&= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger
\end{aligned}$$

Then by Hermiticity

$$\begin{aligned}
\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger &= \hat{B}\hat{A} - \hat{A}\hat{B} \\
&= -[\hat{A}, \hat{B}]
\end{aligned}$$