

(3.2) Show that for the simple harmonic oscillator:

- (a) $[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1}$,
 - (b) $\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = n!\delta_{nm}$,
 - (c) $\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{m,n+1}$,
 - (d) $\langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1}$.
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(a) We have for $r = m + n - 1$

$$\begin{aligned} [\hat{a}, (\hat{a}^\dagger)^n] |m\rangle &= \hat{a}(\hat{a}^\dagger)^n |m\rangle - (\hat{a}^\dagger)^n \hat{a} |m\rangle \\ &= \sqrt{m+n} \left(\prod_{k=1}^n \sqrt{m+k} \right) |r\rangle - \left(\prod_{k=1}^n \sqrt{m-1+k} \right) \sqrt{m} |r\rangle \end{aligned}$$

It follows that

$$[\hat{a}, (\hat{a}^\dagger)^n] |m\rangle = (m+n) \left(\prod_{k=1}^{n-1} \sqrt{m+k} \right) |r\rangle - m \left(\prod_{k=2}^n \sqrt{m-1+k} \right) |r\rangle$$

Noting that

$$\prod_{k=1}^{n-1} \sqrt{m+k} = \prod_{k=2}^n \sqrt{m-1+k}$$

we have

$$[\hat{a}, (\hat{a}^\dagger)^n] |m\rangle = n \left(\prod_{k=1}^{n-1} \sqrt{m+k} \right) |m+n-1\rangle$$

Hence

$$[\hat{a}, (\hat{a}^\dagger)^n] |m\rangle = n(\hat{a}^\dagger)^{n-1} |m\rangle$$

(b) We have

$$\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = \left(\prod_{j=1}^n \sqrt{j} \right) \left(\prod_{k=1}^m \sqrt{k} \right) \langle n|m\rangle$$

Hence

$$\langle 0|\hat{a}^n(\hat{a}^\dagger)^m|0\rangle = \begin{cases} n!, & n = m \\ 0, & n \neq m \end{cases}$$

(c) We have

$$\langle m|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle$$

Hence

$$\langle m|\hat{a}^\dagger|n\rangle = \begin{cases} \sqrt{n+1}, & m = n+1 \\ 0, & m \neq n+1 \end{cases}$$

(d) We have

$$\langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle$$

Hence

$$\langle m|\hat{a}|n\rangle = \begin{cases} \sqrt{n}, & m = n-1 \\ 0, & m \neq n-1 \end{cases}$$