(3.1) For boson operators satisfying

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}\right] = \delta_{\mathbf{pq}} \tag{3.41}$$

show that

$$\frac{1}{\mathcal{V}} \sum_{\mathbf{pq}} \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})) \left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^{\dagger}\right] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{3.42}$$

where $\mathcal{V}$ is the volume of space over which the system is defined. Repeat this for fermion commutator operators.

Substitute (3.41) into the left-hand side of (3.42) and call it I.

$$I = \frac{1}{\mathcal{V}} \sum_{\mathbf{pq}} \exp(i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y}))\delta_{\mathbf{pq}}$$

By the sifting property of $\delta_{\mathbf{pq}}$ we have

$$I = \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \exp(i\mathbf{p}\cdot(\mathbf{x} - \mathbf{y})) \tag{1}$$

Consider the following identity where N is the number of bosons.

$$\sum_{p} \exp(ip(x - y)) = N\delta_{xy} \tag{2}$$

Substitute (2) into (1) to obtain

$$I = \frac{N^3}{\mathcal{V}}\delta_{\mathbf{xy}}$$

Hence (3.42) is true in the limit $N \to \infty$.

$$\lim_{N\to\infty} \frac{N^3}{\mathcal{V}}\delta_{\mathbf{xy}} = \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

Note: The infinity is absorbed by the delta function. Recall that

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Also, see equation (4.6).

$$\frac{1}{\mathcal{V}} \sum_{\mathbf{p}} \exp(i\mathbf{p}\cdot\mathbf{x}) = \delta^{(3)}(\mathbf{x}) \tag{4.6}$$