(2.1) For the one-dimensional harmonic oscillator, show that with creation and annihilation operators defined as in eqns 2.9 and 2.10, $[\hat{a},\hat{a}]=0$, $[\hat{a}^\dagger,\hat{a}^\dagger]=0$, $[\hat{a},\hat{a}^\dagger]=1$ and $\hat{H}=\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)$.

$$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) \tag{2.9}$$

$$\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right) \tag{2.10}$$

We have by elementary algebra

$$[\hat{a},\hat{a}]=\hat{a}\hat{a}-\hat{a}\hat{a}=0 \qquad [\hat{a}^\dagger,\hat{a}^\dagger]=\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\hat{a}^\dagger=0$$

For the commutator $[\hat{a},\hat{a}^\dagger]$ we have

$$\begin{aligned}\hat{a}\hat{a}^\dagger&=\frac{m\omega}{2\hbar}\left(\hat{x}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)+\frac{i}{m\omega}\hat{p}\left(\hat{x}-\frac{i}{m\omega}\hat{p}\right)\right)\\&=\frac{m\omega}{2\hbar}\left(\hat{x}^2-\frac{i}{m\omega}\hat{x}\hat{p}+\frac{i}{m\omega}\hat{p}\hat{x}+\frac{1}{m^2\omega^2}\hat{p}^2\right)\end{aligned}$$

Rewrite as

$$\hat{a}\hat{a}^\dagger=\frac{m\omega}{2\hbar}\hat{x}^2-\frac{i}{2\hbar}[\hat{x},\hat{p}]+\frac{1}{2\hbar m\omega}\hat{p}^2$$

From page 20 just below equation (2.8) we have

$$[\hat{x},\hat{p}]=i\hbar$$

It follows that

$$\hat{a}\hat{a}^\dagger=\frac{m\omega}{2\hbar}\hat{x}^2+\frac{1}{2\hbar m\omega}\hat{p}^2+\frac{1}{2} \tag{1}$$

By similar argument

$$\hat{a}^\dagger\hat{a}=\frac{m\omega}{2\hbar}\hat{x}^2+\frac{1}{2\hbar m\omega}\hat{p}^2-\frac{1}{2} \tag{2}$$

Hence

$$[\hat{a},\hat{a}^\dagger]=\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}=\frac{1}{2}-\left(-\frac{1}{2}\right)=1$$

Multiply equation (2) by $\omega\hbar$ to obtain

$$\omega\hbar\left(\hat{a}^{\dagger}\hat{a}\right) = \frac{m\omega^2}{2}\hat{x}^2 + \frac{1}{2m}\hat{p}^2 - \frac{\omega\hbar}{2}$$

Rewrite as

$$\omega\hbar\left(\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2}\right) = \frac{m\omega^2}{2}\hat{x}^2 + \frac{1}{2m}\hat{p}^2 \tag{3}$$

Consider equation (2.6).

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{2.6}$$

Substitute (3) into (2.6).

$$\hat{H} = \omega\hbar\left(\hat{a}^{\dagger}\hat{a} + \tfrac{1}{2}\right)$$