(16.1) (a) Solve the Schrodinger equation to find the wave functions $\phi_n(x)$ for a particle in a one-dimensional square well defined by $V(x) = 0$ for $0 \le x \le a$ and $V(x) = \infty$ for $x < 0$ and $x > a$.

(b) Show that the retarded Green's function for this particle is given by

$$G^+(n, t_2, t_1) = \theta(t_2 - t_1) e^{-i\left(\frac{n^2\pi^2}{2ma^2}\right)(t_2 - t_1)} \tag{16.39}$$

(c) Find $G^+(n, \omega)$ for the particle.

(a) Let m be the mass of the particle. The Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi_n(x) + V(x)\phi_n(x) = E_n\phi_n(x)$$

In the region $0 \le x \le a$ we have $V(x) = 0$ hence we can write

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi_n(x) = E_n\phi_n(x), \quad 0 \le x \le a \tag{1}$$

Equation (1) has the following well-known solution.

$$\phi_n(x) = A\sin(kx) + B\cos(kx), \quad k = \frac{\sqrt{2mE_n}}{\hbar} \tag{2}$$

For boundary conditions we have $\phi_n(0) = 0$ and $\phi_n(a) = 0$ because there is no possibility of finding the particle outside the well. The boundary condition $\phi_n(0) = 0$ forces $B = 0$ because $\cos(0) = 1$. The boundary condition $\phi_n(a) = 0$ forces kx to be a multiple of π at $x = a$ hence

$$kx = \frac{n\pi x}{a}$$

It follows from the definition of k in (2) that

$$\frac{\sqrt{2mE_n}}{\hbar} = \frac{n\pi}{a}$$

Solve for E_n to obtain

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$$

Hence the solution to (1) is

$$\phi_n(x) = A\sin\left(\frac{n\pi x}{a}\right), \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$$

Normalize the wavefunction. (Note that A can be complex hence $|A|$.)

$$1 = \int_0^a \phi_n^*(x)\phi_n(x)\,dx = \tfrac{1}{2}|A|^2 a$$

Hence

$$|A| = \sqrt{\frac{2}{a}}$$

(b) Let $\psi_n(x,t)$ be the following solution to the time-dependent Schrodinger equation.

$$\psi_n(x,t) = \phi_n(x)\exp\left(-\frac{iE_n t}{\hbar}\right)$$

We want $G^+(n,t_2,t_1)$ such that

$$\psi_n(x,t_2) = G^+(n,t_2,t_1)\psi_n(x,t_1)$$

The $\phi_n(x)$ cancel leaving just the time-dependent exponentials.

$$\exp\left(-\frac{iE_n t_2}{\hbar}\right) = G^+(n,t_2,t_1)\exp\left(-\frac{iE_n t_1}{\hbar}\right)$$

Hence

$$\begin{aligned}G^+(n,t_2,t_1) &= \theta(t_2-t_1)\exp\left(-\frac{iE_n t_2}{\hbar}\right)\exp\left(\frac{iE_n t_1}{\hbar}\right)\\ &= \theta(t_2-t_1)\exp\left(-\frac{iE_n(t_2-t_1)}{\hbar}\right)\end{aligned}$$

(c) Take the Fourier transform of $G^+(n,t,0)$.

$$\begin{aligned}G^+(n,\omega) &= \int_0^\infty G^+(n,t,0)\exp\left(\frac{i(\omega+i\epsilon)t}{\hbar}\right)dt\\ &= \int_0^\infty \exp\left(-\frac{iE_n t}{\hbar}\right)\exp\left(\frac{i(\omega+i\epsilon)t}{\hbar}\right)dt\\ &= \frac{i\hbar}{E_n-\omega-i\epsilon}\exp\left(-\frac{i(E_n-\omega-i\epsilon)t}{\hbar}\right)\Bigg|_0^\infty\\ &= \frac{i\hbar}{\omega-E_n+i\epsilon}\end{aligned}$$