(14.2) *A demonstration that the photon has spin-1, with only two spin polarizations.*

A photon γ propagates with momentum $q^\mu = (|\mathbf{q}|, 0, 0, |\mathbf{q}|)$. Working with a basis where the two transverse photon polarizations are $\epsilon^\mu_{\lambda=1}(q) = (0, 1, 0, 0)$ and $\epsilon^\mu_{\lambda=2}(q) = (0, 0, 1, 0)$, it may be shown, using Noether's theorem, that the operator $\hat{S}^z$, whose eigenvalue is the z-component spin angular momentum of the photon, obeys the commutation relation

$$\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}\lambda}\right] = i\epsilon^{\mu=1*}_\lambda(q)\hat{a}^\dagger_{\mathbf{q}\lambda=2} - i\epsilon^{\mu=2*}_\lambda(q)\hat{a}^\dagger_{\mathbf{q}\lambda=1} \tag{14.36}$$

(i) Define creation operators for the circular polarizations via

$$\begin{aligned} \hat{b}^\dagger_{\mathbf{q}R} &= -\frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{\mathbf{q}1} + i\hat{a}^\dagger_{\mathbf{q}2}\right) \\ \hat{b}^\dagger_{\mathbf{q}L} &= \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{\mathbf{q}1} - i\hat{a}^\dagger_{\mathbf{q}2}\right) \end{aligned} \tag{14.37}$$

Show that

$$\begin{aligned} \left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}R}\right] &= \hat{b}^\dagger_{\mathbf{q}R} \\ \left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}L}\right] &= -\hat{b}^\dagger_{\mathbf{q}L} \end{aligned} \tag{14.38}$$

(ii) Consider the operation of S^z on a state $|\gamma_{\mathbf{q}\lambda}\rangle = \hat{b}_{\mathbf{q}\lambda}|0\rangle$ containing a single photon propagating along z:

$$\hat{S}^z|\gamma_{\mathbf{q}\lambda}\rangle = \hat{S}^z\hat{b}^\dagger_{\mathbf{q}\lambda}|0\rangle, \quad \lambda = R, L \tag{14.39}$$

Use the results of (i) to argue that the projection of the photon spin along its direction of propagation must be $S^z = \pm 1$.

See Bjorken and Drell Chapter 14 for the full version of this argument.

(i) By hypothesis we have

$$\begin{aligned} \epsilon^{\mu=0*}_{\lambda=1}(q) &= 0 \qquad & \epsilon^{\mu=0*}_{\lambda=2}(q) &= 0 \\ \epsilon^{\mu=1*}_{\lambda=1}(q) &= 1 \qquad & \epsilon^{\mu=1*}_{\lambda=2}(q) &= 0 \\ \epsilon^{\mu=2*}_{\lambda=1}(q) &= 0 \qquad & \epsilon^{\mu=2*}_{\lambda=2}(q) &= 1 \\ \epsilon^{\mu=3*}_{\lambda=1}(q) &= 0 \qquad & \epsilon^{\mu=3*}_{\lambda=2}(q) &= 0 \end{aligned}$$

Hence

$$\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}1}\right] = i\epsilon^{\mu=1*}_{\lambda=1}(q)\hat{a}^\dagger_{\mathbf{q}2} - i\epsilon^{\mu=2*}_{\lambda=1}(q)\hat{a}^\dagger_{\mathbf{q}1} = i\hat{a}^\dagger_{\mathbf{q}2}$$
$$\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}2}\right] = i\epsilon^{\mu=1*}_{\lambda=2}(q)\hat{a}^\dagger_{\mathbf{q}2} - i\epsilon^{\mu=2*}_{\lambda=2}(q)\hat{a}^\dagger_{\mathbf{q}1} = -i\hat{a}^\dagger_{\mathbf{q}1}$$

It follows that

$$\left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}R}\right] = -\frac{1}{\sqrt{2}}\Big(\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}1}\right] + i\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}2}\right]\Big) = -\frac{1}{\sqrt{2}}\left(i\hat{a}^\dagger_{\mathbf{q}2} + \hat{a}^\dagger_{\mathbf{q}1}\right) = \hat{b}^\dagger_{\mathbf{q}R}$$
$$\left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}L}\right] = \frac{1}{\sqrt{2}}\Big(\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}1}\right] - i\left[\hat{S}^z, \hat{a}^\dagger_{\mathbf{q}2}\right]\Big) = \frac{1}{\sqrt{2}}\left(i\hat{a}^\dagger_{\mathbf{q}2} - \hat{a}^\dagger_{\mathbf{q}1}\right) = -\hat{b}^\dagger_{\mathbf{q}L}$$

(ii) From part (i) we have the commutators

$$\left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}R}\right] = \hat{b}^\dagger_{\mathbf{q}R}$$
$$\left[\hat{S}^z, \hat{b}^\dagger_{\mathbf{q}L}\right] = -\hat{b}^\dagger_{\mathbf{q}L}$$

It follows that

$$\hat{S}^z\hat{b}^\dagger_{\mathbf{q}R} = \hat{b}^\dagger_{\mathbf{q}R}\hat{S}^z + \hat{b}^\dagger_{\mathbf{q}R}$$
$$\hat{S}^z\hat{b}^\dagger_{\mathbf{q}L} = \hat{b}^\dagger_{\mathbf{q}L}\hat{S}^z - \hat{b}^\dagger_{\mathbf{q}L}$$

Hence we can write

$$\hat{S}^z|\gamma_{\mathbf{q}R}\rangle = \hat{S}^z\hat{b}^\dagger_{\mathbf{q}R}|0\rangle = \left(\hat{b}^\dagger_{\mathbf{q}R}\hat{S}^z + \hat{b}^\dagger_{\mathbf{q}R}\right)|0\rangle$$
$$\hat{S}^z|\gamma_{\mathbf{q}L}\rangle = \hat{S}^z\hat{b}^\dagger_{\mathbf{q}L}|0\rangle = \left(\hat{b}^\dagger_{\mathbf{q}L}\hat{S}^z - \hat{b}^\dagger_{\mathbf{q}L}\right)|0\rangle$$

Noting that $\hat{S}^z|0\rangle = 0$ we obtain

$$\hat{S}^z|\gamma_{\mathbf{q}R}\rangle = \hat{b}^\dagger_{\mathbf{q}R}|0\rangle = |\gamma_{\mathbf{q}R}\rangle$$
$$\hat{S}^z|\gamma_{\mathbf{q}L}\rangle = -\hat{b}^\dagger_{\mathbf{q}L}|0\rangle = -|\gamma_{\mathbf{q}L}\rangle$$

Hence

$$\hat{S}^z|\gamma_{\mathbf{q}\lambda}\rangle = \pm|\gamma_{\mathbf{q}\lambda}\rangle$$

where the eigenvalue is $+1$ for $\lambda = R$ and -1 for $\lambda = L$.