(12.1) Fill in the missing steps of the algebra that led to eqn 12.6.

Consider equations (12.5) and (12.6).

$$\begin{aligned}\hat{\psi}(x) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \left(\hat{a}_p \exp(-ip\cdot x) + \hat{b}_p^\dagger \exp(ip\cdot x)\right) \\ \hat{\psi}(x)^\dagger &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \left(\hat{a}_p^\dagger \exp(ip\cdot x) + \hat{b}_p \exp(-ip\cdot x)\right)\end{aligned} \tag{12.5}$$

$$\begin{aligned}N[\hat{H}] &= \int d^3p\, E_p \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p\right) \\ &= \int d^3p\, E_p \left(\hat{n}_p^{(a)} + \hat{n}_p^{(b)}\right)\end{aligned} \tag{12.6}$$

The "missing steps" are the substitution of (12.5) into the Hamiltonian to obtain (12.6).

The Hamiltonian density $\mathcal{H}$ is given by (12.3).

$$\mathcal{H} = \partial_0 \psi^\dagger(x) \partial_0 \psi(x) + \nabla\psi^\dagger(x) \cdot \nabla\psi(x) + m^2 \psi^\dagger(x)\psi(x) \tag{12.3}$$

From equation (11.24), the Hamiltonian operator is the volume integral of the Hamiltonian density.

$$\hat{H} = \int d^3x\, \mathcal{H}$$

Recall that

$$p\cdot x = E_p t - p_1 x_1 - p_2 x_2 - p_3 x_3, \quad E_p = +\sqrt{p_1^2 + p_2^2 + p_3^2 + m^2}$$

Hence (see equation 11.25)

$$\begin{aligned}\partial_\mu \hat{\psi}(x) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} (-ip_\mu) \left(\hat{a}_p \exp(-ip\cdot x) - \hat{b}_p^\dagger \exp(ip\cdot x)\right) \\ \partial_\mu \hat{\psi}^\dagger(x) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} (ip_\mu) \left(\hat{a}_p^\dagger \exp(ip\cdot x) - \hat{b}_p \exp(-ip\cdot x)\right)\end{aligned}$$

It follows that

$$\begin{aligned}\partial_0 \psi(x) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} (-iE_p) \left(\hat{a}_p \exp(-ip\cdot x) - \hat{b}_p^\dagger \exp(ip\cdot x)\right) \\ \partial_0 \psi^\dagger(x) &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} (iE_p) \left(\hat{a}_p^\dagger \exp(ip\cdot x) - \hat{b}_p \exp(-ip\cdot x)\right)\end{aligned}$$

and

$$\nabla\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \begin{pmatrix} ip_1 \\ ip_2 \\ ip_3 \end{pmatrix} \Big(\hat{a}_p \exp(-ip\cdot x) - \hat{b}_p^\dagger \exp(ip\cdot x)\Big)$$

$$\nabla\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p)^{\frac{1}{2}}} \begin{pmatrix} -ip_1 \\ -ip_2 \\ -ip_3 \end{pmatrix} \Big(\hat{a}_p^\dagger \exp(ip\cdot x) - \hat{b}_p \exp(-ip\cdot x)\Big)$$

Hence

$$\begin{aligned} \hat{H} = \int d^3x\, \mathcal{H} = \frac{1}{(2\pi)^3} \int \frac{d^3x\, d^3p\, d^3q}{(2E_p)^{\frac{1}{2}}(2E_q)^{\frac{1}{2}}} \Big[&(E_pE_q + p_1q_1 + p_2q_2 + p_3q_3) \\ &\times \Big(\hat{a}_p^\dagger \exp(ip\cdot x) - \hat{b}_p \exp(-ip\cdot x)\Big) \Big(\hat{a}_q \exp(-iq\cdot x) - \hat{b}_q^\dagger \exp(iq\cdot x)\Big) \\ &+ m^2 \Big(\hat{a}_p^\dagger \exp(ip\cdot x) + \hat{b}_p \exp(-ip\cdot x)\Big) \Big(\hat{a}_q \exp(-iq\cdot x) + \hat{b}_q^\dagger \exp(iq\cdot x)\Big) \Big] \end{aligned}$$

Rewrite as

$$\begin{aligned} \hat{H} = \frac{1}{(2\pi)^3} \int \frac{d^3x\, d^3p\, d^3q}{(2E_p)^{\frac{1}{2}}(2E_q)^{\frac{1}{2}}} \Big[&(E_pE_q + p_1q_1 + p_2q_2 + p_3q_3 + m^2)(A + D) \\ &- (E_pE_q + p_1q_1 + p_2q_2 + p_3q_3 - m^2)(B + C) \Big] \end{aligned}$$

where

$$\begin{aligned} A &= \hat{a}_p^\dagger \hat{a}_q \exp(i(p-q)\cdot x) \\ B &= \hat{a}_p^\dagger \hat{b}_q^\dagger \exp(i(p+q)\cdot x) \\ C &= \hat{b}_p \hat{a}_q \exp(-i(p+q)\cdot x) \\ D &= \hat{b}_p \hat{b}_q^\dagger \exp(-i(p-q)\cdot x) \end{aligned}$$

Integrate over x using $\int d^3x\, \exp(ip\cdot x) = (2\pi)^3\delta^{(3)}(p)$ to obtain

$$\begin{aligned} \hat{H} = \int \frac{d^3p\, d^3q}{(2E_p)^{\frac{1}{2}}(2E_q)^{\frac{1}{2}}} \Big[&(E_pE_q + p_1q_1 + p_2q_2 + p_3q_3 + m^2)(A' + D') \\ &- (E_pE_q + p_1q_1 + p_2q_2 + p_3q_3 - m^2)(B' + C') \Big] \end{aligned}$$

where

$$A' = \delta(p_1 - q_1)\delta(p_2 - q_2)\delta(p_3 - q_3)\, \hat{a}_p^\dagger \hat{a}_q \exp(i(E_p - E_q)t)$$
$$B' = \delta(p_1 + q_1)\delta(p_2 + q_2)\delta(p_3 + q_3)\, \hat{a}_p^\dagger \hat{b}_q^\dagger \exp(i(E_p + E_q)t)$$
$$C' = \delta(p_1 + q_1)\delta(p_2 + q_2)\delta(p_3 + q_3)\, \hat{b}_p \hat{a}_q \exp(-i(E_p + E_q)t)$$
$$D' = \delta(p_1 - q_1)\delta(p_2 - q_2)\delta(p_3 - q_3)\, \hat{b}_p \hat{b}_q^\dagger \exp(-i(E_p - E_q)t)$$

Integrate over q to obtain

$$\hat{H} = \int \frac{d^3p}{2E_p} \Big[(E_p^2 + p_1^2 + p_2^2 + p_3^2 + m^2) \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p \hat{b}_p^\dagger \right)$$
$$- (E_p^2 - p_1^2 - p_2^2 - p_3^2 - m^2) \left(\hat{a}_p^\dagger \hat{b}_{-p}^\dagger \exp(2iE_p t) + \hat{b}_p \hat{a}_{-p} \exp(-2iE_p t) \right) \Big]$$

Noting that $E_p^2 = p_1^2 + p_2^2 + p_3^2 + m^2$ we have

$$\hat{H} = \int d^3p\, E_p \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p \hat{b}_p^\dagger \right)$$

Apply normal ordering to interchange b and $b^\dagger$.

$$N[\hat{H}] = \int d^3p\, E_p \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p \right)$$