(10.3) For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \tag{10.42}$$

evaluate $T^{\mu\nu}$ and show that T^{00} agrees with what you would expect from the Hamiltonian for this Lagrangian. Show that $\partial_\mu T^{\mu\nu} = 0$. Derive expressions for $P^0 = \int d^3x\, T^{00}$ and $P^i = \int d^3x\, T^{0i}$.

Recall that $\partial_\mu \phi$ is the vector

$$\partial_\mu \phi = \begin{pmatrix} \partial_0 \phi \\ \partial_1 \phi \\ \partial_2 \phi \\ \partial_3 \phi \end{pmatrix}$$

Hence

$$\begin{aligned} (\partial_\mu \phi)^2 &= \partial_\mu \phi \partial^\mu \phi \\ &= (\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2 \\ &= (\partial_0 \phi)^2 - (\nabla \phi)^2 \end{aligned}$$

From the book near equation (10.27) we have

$$T^{\mu\nu} = \Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathcal{L}$$

where Π^μ is the momentum density

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

For the Lagrangian given in (10.42) we have

$$\Pi^0 = \partial_0 \phi$$

It follows from $g^{00} = 1$ that

$$\begin{aligned} T^{00} &= \Pi^0 \partial^0 \phi - \mathcal{L} \\ &= \partial_0 \phi \partial^0 \phi - \mathcal{L} \\ &= \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2\phi^2 \end{aligned}$$

The expected Hamiltonian H is

$$H = p\dot{q} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial_0 \phi - \mathcal{L} = T^{00}$$