

(1.3) Consider the functional $G[f] = \int g(y, f) dy$. Show that

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f} \quad (1.41)$$

Now consider the functional $H[f] = \int g(y, f, f') dy$ and show that

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'} \quad (1.42)$$

where $f' = \partial f / \partial y$. For the functional $J[f] = \int g(y, f, f', f'') dy$ show that

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'} + \frac{d^2}{dx^2} \frac{\partial g}{\partial f''} \quad (1.43)$$

where $f'' = \partial^2 f / \partial y^2$.

Let

$$h = f + \epsilon \delta(y - x) \quad h' = \frac{\partial h}{\partial y} \quad h'' = \frac{\partial^2 h}{\partial y^2}$$

Show (1.41).

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int g(y, h) dy - \int g(y, f) dy \right)$$

Taylor expand the first integrand (ϵ^k vanishes in the limit for all $k > 1$).

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int g(y, f) dy + \int \epsilon \delta(y - x) \frac{\partial g(y, f)}{\partial f} dy - \int g(y, f) dy \right)$$

Hence

$$\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f}$$

Show (1.42).

$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int g(y, h, h') dy - \int g(y, f, f') dy \right)$$

Taylor expand the first integrand.

$$g(y, h, h') = g(y, f, f') + \epsilon \delta(y - x) \frac{\partial g(y, f, f')}{\partial f} + \epsilon \frac{\partial \delta(y - x)}{\partial y} \frac{\partial g(y, f, f')}{\partial f'}$$

Hence

$$\frac{\delta H[f]}{\delta f(x)} = \int \delta(y-x) \frac{\partial g(y, f, f')}{\partial f} dy + \int \frac{\partial \delta(y-x)}{\partial y} \frac{\partial g(y, f, f')}{\partial f'} dy \quad (1)$$

The first integral is solved by (1.41). For the second integral, let

$$u = \frac{\partial g(y, f, f')}{\partial f'} \quad v = \delta(y-x)$$

Then

$$du = \frac{\partial}{\partial y} \frac{\partial g(y, f, f')}{\partial f'} dy \quad dv = \frac{\partial \delta(y-x)}{\partial y} dy$$

Integrate by parts.

$$\begin{aligned} & \int \frac{\partial \delta(y-x)}{\partial y} \frac{\partial g(y, f, f')}{\partial f'} dy \\ &= uv - \int v du \\ &= \frac{\partial g(y, f, f')}{\partial f'} \delta(y-x) - \int \delta(y-x) \frac{\partial}{\partial y} \frac{\partial g(y, f, f')}{\partial f'} dy \\ &= -\frac{\partial}{\partial x} \frac{\partial g(x, f, f')}{\partial f'} \end{aligned} \quad (2)$$

Substitute (1.41) and (2) into (1) to obtain (1.42).

$$\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g(x, f, f')}{\partial f} - \frac{\partial}{\partial x} \frac{\partial g(x, f, f')}{\partial f'}$$

Show (1.43).

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(\int g(y, h, h', h'') dy - \int g(y, f, f', f'') dy \right)$$

Taylor expand the first integrand.

$$\begin{aligned} g(y, h, h', h'') &= g(y, f, f', f'') + \epsilon \delta(y-x) \frac{\partial g(y, f, f', f'')}{\partial f} \\ &+ \epsilon \frac{\partial \delta(y-x)}{\partial y} \frac{\partial g(y, f, f', f'')}{\partial f'} + \epsilon \frac{\partial^2 \delta(y-x)}{\partial y^2} \frac{\partial g(y, f, f', f'')}{\partial f''} \end{aligned}$$

Hence

$$\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \int \delta(y-x) \frac{\partial g(y, f, f', f'')}{\partial f} dy \\
&+ \int \frac{\partial \delta(y-x)}{\partial y} \frac{\partial g(y, f, f', f'')}{\partial f'} dy \\
&+ \int \frac{\partial^2 \delta(y-x)}{\partial y^2} \frac{\partial g(y, f, f', f'')}{\partial f''} dy
\end{aligned} \tag{3}$$

The first and second integrals are solved by (1.42). For the third integral, let

$$u = \frac{\partial g(y, f, f', f'')}{\partial f''} \quad v = \frac{\partial \delta(y-x)}{\partial y}$$

Then

$$du = \frac{\partial}{\partial y} \frac{\partial g(y, f, f', f'')}{\partial f''} dy \quad dv = \frac{\partial^2 \delta(y-x)}{\partial y^2} dy$$

Integrate by parts.

$$\begin{aligned}
&\int \frac{\partial^2 \delta(y-x)}{\partial y^2} \frac{\partial g(y, f, f', f'')}{\partial f''} dy \\
&= uv - \int v du \\
&= \frac{\partial g(y, f, f', f'')}{\partial f''} \frac{\partial \delta(y-x)}{\partial y} - \int \frac{\partial \delta(y-x)}{\partial y} \frac{\partial}{\partial y} \frac{\partial g(y, f, f', f'')}{\partial f''} dy
\end{aligned}$$

As before, the uv term vanishes. For the remaining integral, use integration by parts as in (2) to obtain

$$= \frac{\partial^2}{\partial x^2} \frac{\partial g(x, f, f', f'')}{\partial f''} \tag{4}$$

Substitute (1.42) and (4) into (3) to obtain (1.43).

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g(x, f, f', f'')}{\partial f} - \frac{\partial}{\partial x} \frac{\partial g(x, f, f', f'')}{\partial f'} + \frac{\partial^2}{\partial x^2} \frac{\partial g(x, f, f', f'')}{\partial f''}$$