(1.2) Consider the functionals $H[f] = \int G(x,y)f(y)\,dy$, $I[f] = \int_{-1}^{1} f(x)\,dx$ and $J[f] = \int \left(\frac{\partial f}{\partial y}\right)^2 dy$ of the function f. Find the functional derivatives $\frac{\delta H[f]}{\delta f(z)}$, $\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)}$ and $\frac{\delta J[f]}{\delta f(x)}$.

For the first functional derivative we have

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int G(x,y)\big(f(y)+\epsilon\delta(y-z)\big)\,dy - \int G(x,y)f(y)\,dy\right)$$

The integrals of $G(x,y)f(y)$ cancel and ϵ cancels.

$$\frac{\delta H[f]}{\delta f(z)} = \int G(x,y)\delta(y-z)\,dy = G(x,z)$$

For the second functional derivative we have

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \frac{\delta}{\delta f(x_0)}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{-1}^{1}\big(f(x)+\epsilon\delta(x-x_1)\big)^3\,dx - \int_{-1}^{1} f(x)^3\,dx\right)$$

Terms involving ϵ^2 and ϵ^3 vanish in the limit hence

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \frac{\delta}{\delta f(x_0)}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_{-1}^{1} f(x)^3\,dx + \epsilon\int_{-1}^{1} 3f(x)^2\delta(x-x_1)\,dx - \int_{-1}^{1} f(x)^3\,dx\right)$$

The integrals of $f(x)^3$ cancel and ϵ cancels.

$$\begin{aligned}\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} &= \frac{\delta}{\delta f(x_0)}\int_{-1}^{1} 3f(x)^2\delta(x-x_1)\,dx \\ &= \frac{\delta}{\delta f(x_0)}\begin{cases}3f(x_1)^2 & -1\le x_1\le 1\\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{1}$$

Now do the second derivative.

$$\frac{\delta}{\delta f(x_0)}\left(3f(x_1)^2\right) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(3\big(f(x_1)+\epsilon\delta(x_1-x_0)\big)^2 - 3f(x_1)^2\right)$$

The term involving ϵ^2 vanishes in the limit hence

$$\begin{aligned}\frac{\delta}{\delta f(x_0)}\left(3f(x_1)^2\right) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(3f(x_1)^2+6f(x_1)\epsilon\delta(x_1-x_0)-3f(x_1)^2\right)\\ &= \begin{cases}6f(x_1) & x_0=x_1\\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{2}$$

Substitute (2) into (1) to obtain

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\delta f(x_1)} = \begin{cases}6f(x_1) & x_0=x_1 \text{ and } -1\le x_1\le 1\\ 0 & \text{otherwise}\end{cases}$$

For the third functional derivative we have

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int\left(\frac{\partial f}{\partial y}+\epsilon\frac{\partial\delta(y-x)}{\partial y}\right)^2 dy-\int\left(\frac{\partial f}{\partial y}\right)^2 dy\right)$$

The term involving ϵ^2 vanishes in the limit hence

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int\left(\frac{\partial f}{\partial y}\right)^2 dy+2\epsilon\int\frac{\partial f}{\partial y}\frac{\partial\delta(y-x)}{\partial y}\,dy-\int\left(\frac{\partial f}{\partial y}\right)^2 dy\right)$$

After cancellations

$$\frac{\delta J[f]}{\delta f(x)} = 2\int\frac{\partial f}{\partial y}\frac{\partial\delta(y-x)}{\partial y}\,dy$$

Let

$$u=\frac{\partial f}{\partial y}\qquad v=\delta(y-x)$$

Then

$$du=\frac{\partial^2 f}{\partial y^2}\,dy\qquad dv=\frac{\partial\delta(y-x)}{\partial y}\,dy$$

and

$$\frac{\delta J[f]}{\delta f(x)} = 2\int u\,dv$$

Integrate by parts.

$$\begin{aligned}\frac{\delta J[f]}{\delta f(x)} &= 2uv-2\int v\,du\\ &= 2\frac{\partial f}{\partial y}\delta(y-x)-2\int\delta(y-x)\frac{\partial^2 f}{\partial y^2}\,dy\\ &= -2\frac{\partial^2 f}{\partial x^2}\qquad \text{see equation (1.19)}\end{aligned}$$