8-1. The amplitude to go from any state $\psi(x)$ to another state $\chi(x)$ is the transition amplitude $\langle\chi|1|\psi\rangle$ as defined in equation (7.1). Suppose $\psi(x)$ and $\chi(x)$ are expanded in terms of the orthogonal functions $\phi_n(x)$, the energy solutions to the wave equation associated with the kernel $K(b,a)$, as discussed in section 4-2. Thus

$$\psi(x) = \sum_n \psi_n \phi_n(x), \qquad \chi(x) = \sum_n \chi_n \phi_n(x) \tag{8.23}$$

Using the coefficients ψ_n and χ_n and equation (4.59), show that the transition amplitude can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^*(x_b) K(x_b, T; x_a, 0)\psi(x_a)\, dx_a\, dx_b = \sum_n \chi_n^* \psi_n \exp\left(-\frac{i}{\hbar}E_n T\right) \tag{8.24}$$

This is equation (4.59).

$$K(x_b, t_b; x_a, t_a) = \begin{cases} \sum_{n=1}^{\infty} \phi_n(x_b)\phi_n^*(x_a) \exp\left(-\frac{i}{\hbar}E_n(t_b - t_a)\right) & t_b > t_a \\ 0 & t_b < t_a \end{cases} \tag{4.59}$$

Let

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^*(x_b) K(x_b, T; x_a, 0)\psi(x_a)\, dx_a\, dx_b$$

By (8.23) and (4.59) we have

$$I = \sum_j \sum_n \sum_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_j^* \phi_j^*(x_b)\ \phi_n(x_b)\phi_n^*(x_a) \exp\left(-\frac{i}{\hbar}E_n T\right)\ \psi_k \phi_k(x_a)\, dx_a\, dx_b$$

The integrals for $j, k \neq n$ vanish by orthogonality hence

$$I = \sum_n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_n^* \phi_n^*(x_b)\ \phi_n(x_b)\phi_n^*(x_a) \exp\left(-\frac{i}{\hbar}E_n T\right)\ \psi_n \phi_n(x_a)\, dx_a\, dx_b$$

Rewrite as

$$I = \sum_n \chi_n^* \exp\left(-\frac{i}{\hbar}E_n T\right) \psi_n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_n^*(x_b)\phi_n(x_b)\phi_n^*(x_a)\phi_n(x_a)\, dx_a\, dx_b$$

The remaining integrals are unity by normalization hence

$$I = \sum_n \chi_n^* \exp\left(-\frac{i}{\hbar}E_n T\right) \psi_n$$