

7-6. Show, for a particle moving in three-dimensional space  $x, y, z$ ,

$$\langle (x_{k+1} - x_k)^2 \rangle = \langle (y_{k+1} - y_k)^2 \rangle = \langle (z_{k+1} - z_k)^2 \rangle = -\frac{\hbar\epsilon}{im} \langle 1 \rangle \quad (7.50)$$

$$\begin{aligned} \langle (x_{k+1} - x_k)(y_{k+1} - y_k) \rangle &= \langle (x_{k+1} - x_k)(z_{k+1} - z_k) \rangle \\ &= \langle (y_{k+1} - y_k)(z_{k+1} - z_k) \rangle = 0 \end{aligned} \quad (7.51)$$

To extend (7.33) to three dimensions, change derivative to gradient, hence

$$\langle \nabla_k F \rangle = -\frac{i}{\hbar} \langle F \nabla_k S \rangle \quad (1)$$

where

$$\nabla_k = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

and

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let

$$F = x_k + y_k + z_k$$

Then

$$\begin{aligned} \nabla_k F &= \frac{\partial F}{\partial x_k} \mathbf{i} + \frac{\partial F}{\partial y_k} \mathbf{j} + \frac{\partial F}{\partial z_k} \mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

Hence

$$\langle \nabla_k F \rangle = \langle 1 \rangle \mathbf{i} + \langle 1 \rangle \mathbf{j} + \langle 1 \rangle \mathbf{k} \quad (2)$$

By equation (7.43)

$$\begin{aligned} \left\langle (x_k + y_k + z_k) \frac{\partial S}{\partial x_k} \right\rangle \mathbf{i} &= \left\langle x_k \frac{\partial S}{\partial x_k} \right\rangle \mathbf{i} \\ \left\langle (x_k + y_k + z_k) \frac{\partial S}{\partial y_k} \right\rangle \mathbf{j} &= \left\langle y_k \frac{\partial S}{\partial y_k} \right\rangle \mathbf{j} \\ \left\langle (x_k + y_k + z_k) \frac{\partial S}{\partial z_k} \right\rangle \mathbf{k} &= \left\langle z_k \frac{\partial S}{\partial z_k} \right\rangle \mathbf{k} \end{aligned}$$

Hence

$$\langle F \nabla_k S \rangle = \left\langle x_k \frac{\partial S}{\partial x_k} \right\rangle \mathbf{i} + \left\langle y_k \frac{\partial S}{\partial y_k} \right\rangle \mathbf{j} + \left\langle z_k \frac{\partial S}{\partial z_k} \right\rangle \mathbf{k} \quad (3)$$

By equations (1), (2), and (3)

$$\begin{aligned} \langle 1 \rangle \mathbf{i} &= -\frac{i}{\hbar} \left\langle x_k \frac{\partial S}{\partial x_k} \right\rangle \mathbf{i} \\ \langle 1 \rangle \mathbf{j} &= -\frac{i}{\hbar} \left\langle y_k \frac{\partial S}{\partial y_k} \right\rangle \mathbf{j} \\ \langle 1 \rangle \mathbf{k} &= -\frac{i}{\hbar} \left\langle z_k \frac{\partial S}{\partial z_k} \right\rangle \mathbf{k} \end{aligned}$$

Then by the same arguments in the book for equation (7.49) we have

$$\begin{aligned} \langle (x_{k+1} - x_k)^2 \rangle \mathbf{i} &= -\frac{\hbar \epsilon}{im} \langle 1 \rangle \mathbf{i} \\ \langle (y_{k+1} - y_k)^2 \rangle \mathbf{j} &= -\frac{\hbar \epsilon}{im} \langle 1 \rangle \mathbf{j} \\ \langle (z_{k+1} - z_k)^2 \rangle \mathbf{k} &= -\frac{\hbar \epsilon}{im} \langle 1 \rangle \mathbf{k} \end{aligned}$$

Hence (7.50) is shown to be true.

The action for a particle in three dimensions is

$$S = \int_{t_a}^{t_b} \left( \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \right) dt$$

Hence

$$\nabla_k S = \frac{\partial S}{\partial x_k} \mathbf{i} + \frac{\partial S}{\partial y_k} \mathbf{j} + \frac{\partial S}{\partial z_k} \mathbf{k}$$

where

$$\begin{aligned} \frac{\partial S}{\partial x_k} &= -m \left( \frac{x_{k+1} - x_k}{\epsilon} - \frac{x_k - x_{k-1}}{\epsilon} \right) - \epsilon \frac{\partial V}{\partial x} \Big|_{x_k, y_k, z_k} \\ \frac{\partial S}{\partial y_k} &= -m \left( \frac{y_{k+1} - y_k}{\epsilon} - \frac{y_k - y_{k-1}}{\epsilon} \right) - \epsilon \frac{\partial V}{\partial y} \Big|_{x_k, y_k, z_k} \\ \frac{\partial S}{\partial z_k} &= -m \left( \frac{z_{k+1} - z_k}{\epsilon} - \frac{z_k - z_{k-1}}{\epsilon} \right) - \epsilon \frac{\partial V}{\partial z} \Big|_{x_k, y_k, z_k} \end{aligned}$$