7-3. If

$$F\big(j(\mathbf{r},t)\big) = \exp\left(\frac{1}{2}\int\int\int\int j(\mathbf{r}_1,t_1)j(\mathbf{r}_2,t_2)R(\mathbf{r}_2-\mathbf{r}_1,t_2-t_1)\,d^3\mathbf{r}_2\,dt_2\,d^3\mathbf{r}_1\,dt_1\right)$$

where the integrals extend over all space and time, show that

$$\frac{\delta F}{\delta j(\mathbf{x},s)} = F\int\int j(\mathbf{r},t)\tfrac{1}{2}\big(R(\mathbf{r}-\mathbf{x},t-s)+R(\mathbf{x}-\mathbf{r},s-t)\big)\,d^3\mathbf{r}\,dt \qquad (7.27)$$

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By equation (7.20) to first order we have

$$F\big(j(\mathbf{r},t)+\eta(\mathbf{r},t)\big) - F\big(j(\mathbf{r},t)\big) = \int\int \frac{\delta F}{\delta j(\mathbf{x},s)}\eta(\mathbf{x},s)\,d^3\mathbf{x}\,ds$$

Let

$$I(f_1,f_2) = \frac{1}{2}\int\int\int\int f_1(\mathbf{r}_1,t_1)f_2(\mathbf{r}_2,t_2)R(\mathbf{r}_2-\mathbf{r}_1,t_2-t_1)\,d^3\mathbf{r}_2\,dt_2\,d^3\mathbf{r}_1\,dt_1$$

Then

$$F\big(j(\mathbf{r},t)+\eta(\mathbf{r},t)\big) = F\big(j(\mathbf{r},t)\big)\exp\big(I(j,\eta)\big)\exp\big(I(\eta,j)\big)\exp\big(I(\eta,\eta)\big)$$

Use the approximation $\exp(x)=1+x$ to obtain to first order in $\eta$

$$F\big(j(\mathbf{r},t)+\eta(\mathbf{r},t)\big) = F\big(j(\mathbf{r},t)\big)\big(1+I(j,\eta)+I(\eta,j)\big)$$

Then by equation (7.20)

$$F\big(j(\mathbf{r},t)\big)I(j,\eta) + F\big(j(\mathbf{r},t)\big)I(\eta,j) = \int\int \frac{\delta F}{\delta j(\mathbf{x},s)}\eta(\mathbf{x},s)\,d^3\mathbf{x}\,ds \qquad (1)$$

Substitute (7.27) into (1). The $F\big(j(\mathbf{r},t)\big)$ cancel leaving

$$\begin{aligned} I(j,\eta)+I(\eta,j) = {} & \frac{1}{2}\int\int\int\int j(\mathbf{r},t)R(\mathbf{r}-\mathbf{x},t-s)\eta(\mathbf{x},s)\,d^3\mathbf{r}\,dt\,d^3\mathbf{x}\,ds \\ & + \frac{1}{2}\int\int\int\int j(\mathbf{r},t)R(\mathbf{x}-\mathbf{r},s-t)\eta(\mathbf{x},s)\,d^3\mathbf{r}\,dt\,d^3\mathbf{x}\,ds \end{aligned}$$

which is easily verified.