WKB approximation

Start with the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Rewrite as

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\big(E - V(x)\big)\psi \tag{1}$$

Let ψ be composed of amplitude A and phase ϕ.

$$\psi(x) = A(x)e^{i\phi(x)} \tag{2}$$

It follows that

$$\frac{d\psi}{dx} = \left(\frac{dA}{dx} + iA\frac{d\phi}{dx}\right)e^{i\phi}$$

and

$$\frac{d^2\psi}{dx^2} = \left(\frac{d^2A}{dx^2} + 2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2\right)e^{i\phi} \tag{3}$$

Substitute (2) and (3) into (1) to obtain

$$\left(\frac{d^2A}{dx^2} + 2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2\right)e^{i\phi} = -\frac{2m}{\hbar^2}\big(E - V(x)\big)Ae^{i\phi} \tag{4}$$

Partition (4) as a real equivalence

$$\frac{d^2A}{dx^2} - A\left(\frac{d\phi}{dx}\right)^2 = -\frac{2m}{\hbar^2}\big(E - V(x)\big)A \tag{5}$$

and an imaginary equivalence

$$2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2} = 0 \tag{6}$$

Divide equation (5) by A to obtain

$$\frac{1}{A}\frac{d^2A}{dx^2} - \left(\frac{d\phi}{dx}\right)^2 = -\frac{2m}{\hbar^2}\big(E - V(x)\big)$$

Rewrite as

$$\left(\frac{d\phi}{dx}\right)^2 = \frac{2m}{\hbar^2}\big(E - V(x)\big) + \frac{1}{A}\frac{d^2A}{dx^2}$$

For the circumstance of

$$\left|\frac{2m}{\hbar^2}\big(E - V(x)\big)\right| \gg \left|\frac{1}{A}\frac{d^2A}{dx^2}\right|$$

we can use the approximation

$$\left(\frac{d\phi}{dx}\right)^2 \approx \frac{2m}{\hbar^2}\big(E - V(x)\big)$$

Hence

$$\frac{d\phi}{dx} \approx \pm\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)} \tag{7}$$

and

$$\phi(x) \approx \pm\int \sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\,dx \tag{8}$$

We can rewrite (6) as

$$\frac{i}{A}\frac{d}{dx}\left(A^2\frac{d\phi}{dx}\right) = 0 \tag{9}$$

by noting that

$$\frac{i}{A}\frac{d}{dx}\left(A^2\frac{d\phi}{dx}\right) = \frac{i}{A}\left(2A\frac{dA}{dx}\frac{d\phi}{dx} + A^2\frac{d^2\phi}{dx^2}\right) = 2i\frac{dA}{dx}\frac{d\phi}{dx} + iA\frac{d^2\phi}{dx^2}$$

Multiply by sides of (9) by $-iA$ to obtain

$$\frac{d}{dx}\left(A^2\frac{d\phi}{dx}\right) = 0$$

By antiderivative

$$A^2\frac{d\phi}{dx} = C^2$$

where C^2 is an arbitrary constant. Hence

$$A(x) = C\left(\frac{d\phi}{dx}\right)^{-\frac{1}{2}} \tag{10}$$

Substitute (7) into (10) to obtain

$$A(x) \approx C\left(\pm\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \tag{11}$$

Substitute (8) and (11) into (2) to obtain

$$\psi(x) \approx C\left(\pm\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \exp\left(\pm i\int\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\,dx\right)$$

Rewrite as

$$\psi(x) \approx Ce^{i\theta}\left(\frac{2m}{\hbar^2}\big|E - V(x)\big|\right)^{-\frac{1}{4}} \exp\left(\pm i\int\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\,dx\right) \tag{12}$$

where

$$\theta = \begin{cases} 0 & \text{for case of } \left(+\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \text{ and } E > V(x) \\ -\frac{\pi}{4} & \text{for case of } \left(+\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \text{ and } E < V(x) \\ -\frac{\pi}{2} & \text{for case of } \left(-\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \text{ and } E > V(x) \\ -\frac{3\pi}{4} & \text{for case of } \left(-\sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\right)^{-\frac{1}{2}} \text{ and } E < V(x) \end{cases}$$

The constant $e^{i\theta}$ can be discarded because it cancels in (1) hence

$$\psi(x) \approx C \left(\frac{2m}{\hbar^2}\big|E - V(x)\big|\right)^{-\frac{1}{4}} \exp\left(\pm i \int \sqrt{\frac{2m}{\hbar^2}\big(E - V(x)\big)}\, dx\right)$$

(Ref. "WKB approximation" at physicspages.com)