Tunneling

Let $V(x)$ be the barrier potential

$$V(x) = \begin{cases} 0 & x < -a \\ V_0 & -a \le x \le a \\ 0 & x > a \end{cases}$$

Find tunneling probability T for a particle with mass m and energy $E < V_0$.

§1

We have the following Schrodinger equations, one for each region of $V(x)$.

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_1 &= E\psi_1 && x \le -a \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_2 + V_0\psi_2 &= E\psi_2 && -a \le x \le a \\ -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_3 &= E\psi_3 && x \ge a \end{aligned}$$

The solutions are

$$\begin{aligned} \psi_1(x) &= A\exp(ikx) + B\exp(-ikx) \\ \psi_2(x) &= C\exp(i\beta x) + D\exp(-i\beta x) \\ \psi_3(x) &= F\exp(ikx) \end{aligned}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \beta = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Boundary conditions at $x = -a$.

$$\begin{aligned} \psi_1(-a) &= \psi_2(-a) \\ \psi_1'(-a) &= \psi_2'(-a) \end{aligned}$$

Boundary conditions at $x = a$.

$$\begin{aligned} \psi_2(a) &= \psi_3(a) \\ \psi_2'(a) &= \psi_3'(a) \end{aligned}$$

From boundary conditions for ψ_2 and ψ_3 we have

$$C\exp(i\beta a) + D\exp(-i\beta a) = F\exp(ikx) \tag{1}$$

and

$$i\beta C\exp(i\beta a) - i\beta D\exp(-i\beta a) = ikF\exp(ikx) \tag{2}$$

Add $i\beta$ times (1) to (2) to obtain

$$2i\beta C\exp(i\beta a) = (i\beta + ik)F\exp(ika)$$

Hence

$$C = \frac{(\beta + k)F\exp(ika - i\beta a)}{2\beta}$$

Add minus $i\beta$ times (1) to (2) to obtain

$$-2i\beta D\exp(-i\beta a) = (-i\beta + ik)F\exp(ika)$$

Hence

$$D = \frac{(\beta - k)F\exp(ika + i\beta a)}{2\beta}$$

From boundary conditions for ψ_1 and ψ_2 we have

$$A\exp(-ika) + B\exp(ika) = C\exp(-i\beta a) + D\exp(i\beta a) \tag{3}$$

and

$$ikA\exp(-ika) - ikB\exp(ika) = i\beta C\exp(-i\beta a) - i\beta D\exp(i\beta a) \tag{4}$$

Add ik times (3) to (4) to obtain

$$2ikA\exp(-ika) = (ik + i\beta)C\exp(-i\beta a) + (ik - i\beta)D\exp(i\beta a)$$

Hence

$$A = \frac{(k + \beta)C\exp(ika - i\beta a)}{2k} + \frac{(k - \beta)D\exp(ika + i\beta a)}{2k}$$

Substitute for C and D.

$$A = \frac{(k + \beta)^2 F\exp[2ia(k - \beta)]}{4k\beta} - \frac{(k - \beta)^2 F\exp[2ia(k + \beta)]}{4k\beta}$$

Hence

$$\frac{A}{F} = \frac{(k + \beta)^2\exp[2ia(k - \beta)]}{4k\beta} - \frac{(k - \beta)^2\exp[2ia(k + \beta)]}{4k\beta} \tag{5}$$

§2

Substituting for k and β we have for tunneling probability T

$$T^{-1} = \frac{A}{F}\left(\frac{A}{F}\right)^* = 1 + \frac{1}{8}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\left[\cos\left(\frac{4a}{\hbar}\sqrt{2m(E - V_0)}\right) - 1\right] \tag{6}$$

Transform cos to sinh.

$$T^{-1} = 1 + \frac{1}{4}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\sinh^2\left(\frac{2ia}{\hbar}\sqrt{2m(E - V_0)}\right) \tag{7}$$

Cancel the imaginary unit by swapping E and V_0.

$$T^{-1} = 1 + \frac{1}{4}\left(\frac{E}{V_0 - E} + \frac{V_0}{E} + 1\right)\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)$$

Equivalently

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right) \tag{8}$$

§3

Check dimensions. Square root of kilogram joules is momentum.

$$\frac{2a}{\hbar}\sqrt{2m(V_0 - E)} = \frac{[\mathrm{m}]}{[\mathrm{J\,s}]}\sqrt{[\mathrm{kg}]\,[\mathrm{J}]} = [1] \tag{9}$$

Eigenmath script