Time evolution operator

Find operator $U(t, t_0)$ that evolves a quantum state from time t_0 to time t.

$$\psi(x,t) = U(t,t_0)\psi(x,t_0)$$

Start with the Schrodinger equation.

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(x,t)\psi(x,t)$$

Substitute for $\psi(x,t)$.

$$i\hbar\frac{\partial}{\partial t}\Big[U(t,t_0)\psi(x,t_0)\Big] = H(x,t)U(t,t_0)\psi(x,t_0)$$

Expand the left-hand side by the product rule.

$$i\hbar\frac{\partial}{\partial t}U(t,t_0)\psi(x,t_0) + \boxed{i\hbar U(t,t_0)\frac{\partial}{\partial t}\psi(x,t_0)} = H(x,t)U(t,t_0)\psi(x,t_0)$$

The boxed term drops out due to $\partial\psi(x,t_0)/\partial t = 0$.

$$i\hbar\frac{\partial}{\partial t}U(t,t_0)\psi(x,t_0) = H(x,t)U(t,t_0)\psi(x,t_0)$$

By operator equivalence

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = H(x,t)U(t,t_0)$$

Rearrange as

$$\frac{1}{U(t,t_0)}\,\partial U(t,t_0) = -\frac{i}{\hbar}H(x,t)\,\partial t$$

Integrate both sides.

$$\int_{t_0}^{t}\frac{1}{U(\tau,t_0)}\,dU(\tau,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t}H(x,\tau)\,d\tau$$

Solve the first integral.

$$\log U(t,t_0) - \boxed{\log U(t_0,t_0)} = -\frac{i}{\hbar}\int_{t_0}^{t}H(x,\tau)\,d\tau$$

By $U(t_0,t_0) = 1$ the boxed term vanishes.

$$\log U(t,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t}H(x,\tau)\,d\tau$$

Exponential of both sides.

$$U(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}H(x,\tau)\,d\tau\right) \tag{1}$$

If H is time-independent then the integral can be replaced by $(t-t_0)H(x)$.

$$U(t,t_0) = \exp\left(-\frac{i}{\hbar}H(x)\int_{t_0}^{t}d\tau\right) = \exp\left(-\frac{i}{\hbar}(t-t_0)H(x)\right)$$

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