

## Stern-Gerlach 2

From the previous section we have the following Schrödinger equations for a Stern-Gerlach experiment.

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_1 + \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_1 &= i\hbar \frac{\partial}{\partial t} \psi_1 \\ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_2 - \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_2 &= i\hbar \frac{\partial}{\partial t} \psi_2 \end{aligned}$$

We now seek solutions for  $\psi_1$  and  $\psi_2$ .

An online paper<sup>1</sup> provides the following solutions.

$$\begin{aligned} \psi_1 &= \text{Ai} \left[ \left( \frac{\alpha e}{2\hbar} \right)^{1/3} \frac{\alpha e \hbar}{8m^2} t^2 + \left( \frac{\alpha e}{2\hbar} \right)^{1/3} z \right] \exp \left( -\frac{ie\alpha z t}{4m} \right) \exp \left( -\frac{ieB_0 t}{2m} \right) \\ \psi_2 &= \text{Ai} \left[ \left( \frac{\alpha e}{2\hbar} \right)^{1/3} \frac{\alpha e \hbar}{8m^2} t^2 - \left( \frac{\alpha e}{2\hbar} \right)^{1/3} z \right] \exp \left( +\frac{ie\alpha z t}{4m} \right) \exp \left( +\frac{ieB_0 t}{2m} \right) \end{aligned}$$

Let us try verifying  $\psi_1$  and  $\psi_2$  with a Taylor series expansion of the Airy function  $\text{Ai}(x)$ .

We have

$$\text{Ai}(x) \approx \sum_{k=0}^n a_k x^k$$

where

$$\begin{aligned} a_0 &= \frac{1}{3^{2/3} \Gamma(2/3)} \\ a_1 &= -\frac{1}{3^{1/3} \Gamma(1/3)} \\ a_2 &= 0 \\ a_{n+3} &= \frac{a_n}{(n+3)(n+2)} \end{aligned}$$

For  $n = 9$  we obtain

$$\begin{aligned} \text{Ai}(x) &\approx 2.73941 \times 10^{-5} x^9 - 0.000513531 x^7 + 0.00197238 x^6 \\ &\quad - 0.0215683 x^4 + 0.0591713 x^3 - 0.258819 x + 0.355028 \end{aligned}$$

Calculate  $\psi_1$  and  $\psi_2$  using the  $\text{Ai}(x)$  approximation.

To verify  $\psi_1$  and  $\psi_2$ , calculate departures from equality of the Schrödinger equations.

$$\begin{aligned} \epsilon_1 &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_1 + \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_1 - i\hbar \frac{\partial}{\partial t} \psi_1 \\ \epsilon_2 &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi_2 - \frac{e\hbar}{2m} (B_0 + \alpha z) \psi_2 - i\hbar \frac{\partial}{\partial t} \psi_2 \end{aligned}$$

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<sup>1</sup> “Construction of Exact Solutions for the Stern-Gerlach Effect” by Bulnes and Oliveira.

After calculating  $\epsilon_1$  and  $\epsilon_2$ , cancel exponentials and zero out factors  $t^n$  and  $z^m$  where  $n \geq 8$  and  $m \geq 5$ .

The results are

$$\begin{aligned}\epsilon_1 = & -1.69407 \times \boxed{10^{-20}} \frac{\alpha^{16/3} e^{16/3} \hbar^{8/3} t^6 z^2}{m^7} + 1.69407 \times \boxed{10^{-21}} \frac{\alpha^5 e^5 \hbar^3 t^6 z}{m^7} \\ & - 1.35525 \times \boxed{10^{-19}} \frac{\alpha^{13/3} e^{13/3} \hbar^{5/3} t^4 z^3}{m^5} - 4.33681 \times \boxed{10^{-19}} \frac{\alpha^{10/3} e^{10/3} \hbar^{2/3} t^2 z^4}{m^3} \\ & - 3.46945 \times \boxed{10^{-18}} \frac{\alpha^{7/3} e^{7/3} \hbar^{5/3} t^2 z}{m^3} - 1.38778 \times \boxed{10^{-17}} \frac{\alpha^{4/3} e^{4/3} \hbar^{2/3} z^2}{m} \\ & - 1.0842 \times \boxed{10^{-19}} \frac{i \alpha^{13/3} e^{13/3} \hbar^{8/3} t^5 z}{m^6} + 8.67362 \times \boxed{10^{-19}} \frac{i \alpha^{10/3} e^{10/3} \hbar^{5/3} t^3 z^2}{m^4}\end{aligned}$$

and

$$\begin{aligned}\epsilon_2 = & -1.69407 \times \boxed{10^{-20}} \frac{\alpha^{16/3} e^{16/3} \hbar^{8/3} t^6 z^2}{m^7} - 1.69407 \times \boxed{10^{-21}} \frac{\alpha^5 e^5 \hbar^3 t^6 z}{m^7} \\ & + 1.35525 \times \boxed{10^{-19}} \frac{\alpha^{13/3} e^{13/3} \hbar^{5/3} t^4 z^3}{m^5} - 4.33681 \times \boxed{10^{-19}} \frac{\alpha^{10/3} e^{10/3} \hbar^{2/3} t^2 z^4}{m^3} \\ & + 3.46945 \times \boxed{10^{-18}} \frac{\alpha^{7/3} e^{7/3} \hbar^{5/3} t^2 z}{m^3} - 1.38778 \times \boxed{10^{-17}} \frac{\alpha^{4/3} e^{4/3} \hbar^{2/3} z^2}{m} \\ & + 1.0842 \times \boxed{10^{-19}} \frac{i \alpha^{13/3} e^{13/3} \hbar^{8/3} t^5 z}{m^6} + 8.67362 \times \boxed{10^{-19}} \frac{i \alpha^{10/3} e^{10/3} \hbar^{5/3} t^3 z^2}{m^4}\end{aligned}$$

The numerical values are round off errors hence  $\epsilon_1 = \epsilon_2 = 0$  and  $\psi_1$  and  $\psi_2$  are confirmed as solutions.

Eigenmath script

## Notes

1. Wavefunctions  $\psi_1$  and  $\psi_2$  given above are unnormalized and dimensionless. Recall that

$$\int \psi^* \psi \, dx = 1$$

Hence  $\psi_1$  and  $\psi_2$  must be multiplied by a normalization factor with dimension inverse square root of length to cancel with  $dx$ .

2. In SI units

$$\begin{aligned}\hbar &= [\text{J s}] \\ e &= [\text{A s}] \\ B_0 &= [\text{N A}^{-1} \text{ m}^{-1}] \\ \alpha &= [\text{N A}^{-1} \text{ m}^{-2}]\end{aligned}$$

Hence

$$\begin{aligned}\frac{e\hbar}{2m}(B_0 + \alpha z) &= [\text{J}] \\ \left(\frac{\alpha e}{2\hbar}\right)^{1/3} &= [\text{m}^{-1}] \\ \frac{\alpha e \hbar}{8m^2} &= [\text{m s}^{-2}]\end{aligned}$$

As required by the Airy function, these products are dimensionless.

$$\begin{aligned}\left(\frac{\alpha e}{2\hbar}\right)^{1/3} \frac{\alpha e \hbar}{8m^2} t^2 &= [1] \\ \left(\frac{\alpha e}{2\hbar}\right)^{1/3} z &= [1]\end{aligned}$$

As required by the exponential function, these products are dimensionless.

$$\begin{aligned}\frac{e\alpha z t}{4m} &= [1] \\ \frac{eB_0 t}{2m} &= [1]\end{aligned}$$