Stern-Gerlach 1

From problem 4.74 of Griffiths and Schroeter.

The Hamiltonian for a Stern-Gerlach experiment is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{e}{m}\mathbf{B}\cdot\mathbf{S}$$

Given vectors

$$\mathbf{B} = -\alpha x \begin{pmatrix}1\\0\\0\end{pmatrix} + (B_0 + \alpha z)\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \mathbf{S} = \frac{\hbar}{2}\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix}$$

we have the dot product

$$\frac{e}{m}\mathbf{B}\cdot\mathbf{S} = -\frac{e\hbar}{2m}\alpha x\sigma_x + \frac{e\hbar}{2m}(B_0+\alpha z)\sigma_z$$

Let Ψ be the spin state vector

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

where ψ_1 and ψ_2 are wavefunctions for spin up and down.

Plug H and Ψ into the Schrödinger equation to obtain

$$-\frac{\hbar^2}{2m}\nabla^2\Psi - \frac{e\hbar}{2m}\alpha x\sigma_x\Psi + \frac{e\hbar}{2m}(B_0+\alpha z)\sigma_z\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Noting that

$$\sigma_x\Psi = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Psi = \begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}$$

and

$$\sigma_z\Psi = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Psi = \begin{pmatrix}\psi_1\\ -\psi_2\end{pmatrix}$$

we have

$$-\frac{\hbar^2}{2m}\nabla^2\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} - \frac{e\hbar}{2m}\alpha x\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix} + \frac{e\hbar}{2m}(B_0+\alpha z)\begin{pmatrix}\psi_1\\ -\psi_2\end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

In component form

$$\begin{aligned}
-\frac{\hbar^2}{2m}\nabla^2\psi_1 - \boxed{\frac{e\hbar}{2m}\alpha x\psi_2} + \frac{e\hbar}{2m}(B_0+\alpha z)\psi_1 &= i\hbar\frac{\partial}{\partial t}\psi_1\\
-\frac{\hbar^2}{2m}\nabla^2\psi_2 - \boxed{\frac{e\hbar}{2m}\alpha x\psi_1} - \frac{e\hbar}{2m}(B_0+\alpha z)\psi_2 &= i\hbar\frac{\partial}{\partial t}\psi_2
\end{aligned} \tag{1}$$

We now seek to eliminate the cross terms shown in boxes.

Let

$$\psi_1 = \exp\left(-\frac{ieB_0t}{2m}\right)\phi_1$$
$$\psi_2 = \exp\left(+\frac{ieB_0t}{2m}\right)\phi_2$$

Then

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_1 &= \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1 + \frac{e\hbar}{2m}B_0\psi_1 \\ i\hbar\frac{\partial}{\partial t}\psi_2 &= \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2 - \frac{e\hbar}{2m}B_0\psi_2 \end{aligned} \tag{2}$$

Substitute into (1) to obtain

$$-\frac{\hbar^2}{2m}\nabla^2\psi_1 - \frac{e\hbar}{2m}\alpha x\psi_2 + \frac{e\hbar}{2m}(B_0+\alpha z)\psi_1 = \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1 + \frac{e\hbar}{2m}B_0\psi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}\alpha x\psi_1 - \frac{e\hbar}{2m}(B_0+\alpha z)\psi_2 = \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2 - \frac{e\hbar}{2m}B_0\psi_2$$

The B_0 terms cancel.

$$-\frac{\hbar^2}{2m}\nabla^2\psi_1 - \frac{e\hbar}{2m}\alpha x\psi_2 + \frac{e\hbar}{2m}\alpha z\psi_1 = \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}\alpha x\psi_1 - \frac{e\hbar}{2m}\alpha z\psi_2 = \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\phi_2$$

Multiply through by the conjugate of the respective exponential (note $\psi \to \phi$).

$$-\frac{\hbar^2}{2m}\nabla^2\phi_1 - \exp\left(+\frac{ieB_0t}{m}\right)\frac{e\hbar}{2m}\alpha x\phi_2 + \frac{e\hbar}{2m}\alpha z\phi_1 = i\hbar\frac{\partial}{\partial t}\phi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\phi_2 - \exp\left(-\frac{ieB_0t}{m}\right)\frac{e\hbar}{2m}\alpha x\phi_1 - \frac{e\hbar}{2m}\alpha z\phi_2 = i\hbar\frac{\partial}{\partial t}\phi_2$$

We now simply discard the cross terms with the argument that the exponential factors oscillate causing the cross terms to vanish on average.

$$\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\phi_1 + \frac{e\hbar}{2m}\alpha z\phi_1 &= i\hbar\frac{\partial}{\partial t}\phi_1 \\ -\frac{\hbar^2}{2m}\nabla^2\phi_2 - \frac{e\hbar}{2m}\alpha z\phi_2 &= i\hbar\frac{\partial}{\partial t}\phi_2 \end{aligned} \tag{3}$$

Now do the reverse. Let

$$\phi_1 = \exp\left(+\frac{ieB_0t}{2m}\right)\psi_1$$
$$\phi_2 = \exp\left(-\frac{ieB_0t}{2m}\right)\psi_2$$

Then

$$i\hbar\frac{\partial}{\partial t}\phi_1 = \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\psi_1 - \frac{e\hbar}{2m}B_0\phi_1$$
$$i\hbar\frac{\partial}{\partial t}\phi_2 = \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\psi_2 + \frac{e\hbar}{2m}B_0\phi_2$$

Substitute into (3) to obtain

$$-\frac{\hbar^2}{2m}\nabla^2\phi_1 + \frac{e\hbar}{2m}\alpha z\phi_1 = \exp\left(+\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\psi_1 - \frac{e\hbar}{2m}B_0\phi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\phi_2 - \frac{e\hbar}{2m}\alpha z\phi_2 = \exp\left(-\frac{ieB_0t}{2m}\right)i\hbar\frac{\partial}{\partial t}\psi_2 + \frac{e\hbar}{2m}B_0\phi_2$$

Multiply through by the conjugate of the respective exponential (note $\phi \to \psi$).

$$-\frac{\hbar^2}{2m}\nabla^2\psi_1 + \frac{e\hbar}{2m}\alpha z\psi_1 = i\hbar\frac{\partial}{\partial t}\psi_1 - \frac{e\hbar}{2m}B_0\psi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}\alpha z\psi_2 = i\hbar\frac{\partial}{\partial t}\psi_2 + \frac{e\hbar}{2m}B_0\psi_2$$

Rewrite as

$$-\frac{\hbar^2}{2m}\nabla^2\psi_1 + \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_1 = i\hbar\frac{\partial}{\partial t}\psi_1$$
$$-\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{e\hbar}{2m}(B_0 + \alpha z)\psi_2 = i\hbar\frac{\partial}{\partial t}\psi_2$$

The result is identical to (1) with the boxed terms discarded.

Eigenmath script