Spontaneous emission rate

What is the spontaneous emission rate for hydrogen state $2p$?

The wave function for hydrogen is

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi)$$

where

$$R_{nl}(r) = \frac{2}{n^2}\left(\frac{(n-l-1)!}{(n+l)!}\right)^{1/2}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)\exp\left(-\frac{r}{na_0}\right)a_0^{-3/2}$$

$$L_n^m(x) = (n+m)!\sum_{k=0}^{n}\frac{(-x)^k}{(n-k)!(m+k)!k!}$$

$$Y_{lm}(\theta,\phi) = (-1)^m\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}P_l^m(\cos\theta)\exp(im\phi)$$

$$P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2\mu} \approx 0.529\times 10^{-10}\text{ meter}$$

The state $2p$ means that $n=2$ and $l=1$. For $l=1$ there are three ways to choose m hence all of the following processes correspond to the transition $2p \to 1s$. It turns out that all three processes have the same transition rate.

$$\left.\begin{matrix}\psi_{2,1,1}\\ \psi_{2,1,0}\\ \psi_{2,1,-1}\end{matrix}\right\} \to \psi_{100} + \text{photon}$$

The spontaneous emission rate is

$$A_{21} = \frac{e^2}{3\pi\varepsilon_0\hbar c^3}\omega_{21}^3|r_{21}|^2 \tag{1}$$

where

$$\omega_{21} = \frac{E_2-E_1}{\hbar},\quad E_n = -\frac{e^2}{8\pi\varepsilon_0 a_0 n^2}$$

$$|r_{21}|^2 = |x_{21}|^2 + |y_{21}|^2 + |z_{21}|^2$$

$$x_{21} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} x f_{21}\,dV,\quad y_{21} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} y f_{21}\,dV,\quad z_{21} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} z f_{21}\,dV$$

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

$$f_{21} = \psi_{100}^*\psi_{210} = \frac{r\cos\theta}{4\sqrt{2}\pi a_0^4}\exp\left(-\frac{3r}{2a_0}\right)$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

The integrals are

$$x_{21} = 0, \quad y_{21} = 0, \quad z_{21} = \frac{2^7}{3^5}\sqrt{2}a_0$$

hence

$$|r_{21}|^2 = |z_{21}|^2 = \frac{2^{15}}{3^{10}}a_0^2 = \frac{32768}{59049}a_0^2$$

By equation (1) the spontaneous emission rate is

$$A_{21} = 6.26 \times 10^8 \,\text{second}^{-1}$$

The mean interval is

$$\frac{1}{A_{21}} = 1.60 \times 10^{-9} \,\text{second}$$