Spin operator table

Verify the following spin operator table[1].

$$
\begin{array}{crrrr}
 & |uu\rangle & |ud\rangle & |du\rangle & |dd\rangle \\
\\
\sigma_z & |uu\rangle & |ud\rangle & -|du\rangle & -|dd\rangle \\
\sigma_x & |du\rangle & |dd\rangle & |uu\rangle & |ud\rangle \\
\sigma_y & i|du\rangle & i|dd\rangle & -i|uu\rangle & -i|ud\rangle \\
\tau_z & |uu\rangle & -|ud\rangle & |du\rangle & -|dd\rangle \\
\tau_x & |ud\rangle & |uu\rangle & |dd\rangle & |du\rangle \\
\tau_y & i|ud\rangle & -i|uu\rangle & i|dd\rangle & -i|du\rangle
\end{array}
$$

Table elements are final states. For example,

$$\sigma_z|dd\rangle = -|dd\rangle$$

For single spins we have

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For a system of two spins we use the Kronecker product of single spins.

$$
\begin{aligned}
|uu\rangle &= |u\rangle \otimes |u\rangle \\
|ud\rangle &= |u\rangle \otimes |d\rangle \\
|du\rangle &= |d\rangle \otimes |u\rangle \\
|dd\rangle &= |d\rangle \otimes |d\rangle
\end{aligned}
$$

Hence

$$|uu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The spin operators for single spins are

$$\sigma_z = \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

For a system of two spins, the σ operators operate on the first spin and the τ operators operate on the second spin.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\tau_z = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

[1] "Quantum Mechanics" by Susskind and Friedman, p. 350.