Small rotation

Let J_1, J_2, and J_3 be rotation operators.

$$J_1 = \frac{1}{\hbar} L_1, \quad J_2 = \frac{1}{\hbar} L_2, \quad J_3 = \frac{1}{\hbar} L_3$$

Let U be the unitary transformation

$$U = 1 - i\epsilon J_3 - \tfrac{1}{2}\epsilon^2 {J_3}^2$$

1. Show that to order ϵ^2

$$\begin{aligned}
U^{-1} X_1 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) X_1 - \epsilon X_2 \\
U^{-1} X_2 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) X_2 + \epsilon X_1 \\
U^{-1} X_3 U &= X_3
\end{aligned}$$

2. Show that to order ϵ^2

$$\begin{aligned}
U^{-1} P_1 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) P_1 - \epsilon P_2 \\
U^{-1} P_2 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) P_2 + \epsilon P_1 \\
U^{-1} P_3 U &= P_3
\end{aligned}$$

3. Show that to order ϵ^2

$$\begin{aligned}
U^{-1} L_1 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) L_1 - \epsilon L_2 \\
U^{-1} L_2 U &= \left(1 - \tfrac{1}{2}\epsilon^2\right) L_2 + \epsilon L_1 \\
U^{-1} L_3 U &= L_3
\end{aligned}$$

4. Show that to order ϵ^2

$$U^{-1} H U = H$$

where

$$H = \frac{1}{2m}\left({P_1}^2 + {P_2}^2 + {P_3}^2\right)$$