

Sine perturbation

Let $\Psi(\mathbf{r}, t)$ be the following wave function for a two state system.

$$\Psi(\mathbf{r}, t) = \psi_a(\mathbf{r})c_a(t) \exp(-\frac{i}{\hbar}E_a t) + \psi_b(\mathbf{r})c_b(t) \exp(-\frac{i}{\hbar}E_b t)$$

Let $\hat{H}(\mathbf{r}, t)$ be the Hamiltonian

$$\hat{H}(\mathbf{r}, t) = \hat{H}_0(\mathbf{r}) + \hat{H}_1(\mathbf{r}, t)$$

where

$$\hat{H}_0\psi_a = E_a\psi_a, \quad \hat{H}_0\psi_b = E_b\psi_b, \quad \hat{H}_0\Psi = (E_a + E_b)\Psi$$

It was shown that to first order

$$c_b(t) = -\frac{i}{\hbar} \int_0^t \langle \psi_b | \hat{H}_1 | \psi_a \rangle \exp(i\omega_0 t') dt', \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Let $\hat{H}_1(\mathbf{r}, t)$ be the perturbation

$$\hat{H}_1(\mathbf{r}, t) = \hat{V}(\mathbf{r}) \cos(\omega t)$$

Then by substitution

$$c_b(t) = -\frac{i}{\hbar} \langle \psi_b | \hat{V} | \psi_a \rangle \int_0^t \cos(\omega t') \exp(i\omega_0 t') dt'$$

Solve the integral.

$$\int_0^t \cos(\omega t') \exp(i\omega_0 t') dt' = -\frac{i}{2} \left[\frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} + \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \right] \quad (1)$$

Hence

$$c_b(t) = -\frac{\langle \psi_b | \hat{V} | \psi_a \rangle}{2\hbar} \left[\frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega} + \frac{\exp(i(\omega_0 + \omega)t) - 1}{\omega_0 + \omega} \right] \quad (2)$$

Discard the second term as an approximation since the first term dominates for $\omega \approx \omega_0$.

$$c_b(t) = -\frac{\langle \psi_b | \hat{V} | \psi_a \rangle}{2\hbar} \frac{\exp(i(\omega_0 - \omega)t) - 1}{\omega_0 - \omega}$$

Coefficient $c_b(t)$ can also be written as

$$c_b(t) = -\frac{i}{\hbar} \langle \psi_b | \hat{V} | \psi_a \rangle \frac{\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{\omega_0 - \omega} \exp\left(\frac{i}{2}(\omega_0 - \omega)t\right) \quad (3)$$

Hence the transition probability is

$$P_{a \rightarrow b}(t) = |c_b(t)|^2 = \frac{|\langle \psi_b | \hat{V} | \psi_a \rangle|^2}{\hbar^2} \frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2} \quad (4)$$