# Schrodinger from Lagrangian 1

Derive the Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi$$

from the Lagrangian

$$L(\dot{x},x,t)=\frac{m\dot{x}^2}{2}-V(x,t)$$

Start with the path integral for an action $S$.

$$\psi(x_b,t_b)=C\int_{-\infty}^{\infty}\exp\left(\frac{i}{\hbar}S(b,a)\right)\psi(x_a,t_a)\,dx_a$$

For a small time interval $\epsilon=t_b-t_a$ we can use the approximation

$$S=\epsilon L$$

and write the path integral as

$$\psi(x_b,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left[\frac{i}{\hbar}\epsilon L\left(\frac{x_b-x_a}{\epsilon},\frac{x_b+x_a}{2},t\right)\right]\psi(x_a,t)\,dx_a$$

Substitute for $L$.

$$\psi(x_b,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left[\frac{im(x_b-x_a)^2}{2\hbar\epsilon}-\frac{i}{\hbar}\epsilon V\left(\frac{x_b+x_a}{2},t\right)\right]\psi(x_a,t)\,dx_a$$

Let

$$x_a=x_b+\eta,\quad dx_a=d\eta$$

and write

$$\psi(x_b,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left[\frac{im\eta^2}{2\hbar\epsilon}-\frac{i}{\hbar}\epsilon V\left(x_b+\frac{\eta}{2},t\right)\right]\psi(x_b+\eta,t)\,d\eta$$

Substitute $x$ for $x_b$.

$$\psi(x,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left[\frac{im\eta^2}{2\hbar\epsilon}-\frac{i}{\hbar}\epsilon V\left(x+\frac{\eta}{2},t\right)\right]\psi(x+\eta,t)\,d\eta$$

Because the exponential is highly oscillatory for large $\eta$, most of the contribution to the integral is from small $\eta$. Hence use the approximation $x+\frac{1}{2}\eta\approx x$ for small $\eta$.

$$\psi(x,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left(\frac{im\eta^2}{2\hbar\epsilon}-\frac{i}{\hbar}\epsilon V(x,t)\right)\psi(x+\eta,t)\,d\eta$$

Use the approximation $\exp(y)\approx 1+y$ for the exponential of $V$.

$$\psi(x,t+\epsilon)=C\int_{-\infty}^{\infty}\exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(1-\frac{i}{\hbar}\epsilon V(x,t)\right)\psi(x+\eta,t)\,d\eta$$

Expand $\psi(x+\eta,t)$ as the power series

$$\psi(x+\eta,t) \approx \psi(x,t) + \eta\frac{\partial\psi}{\partial x} + \frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}$$

to obtain

$$\psi(x,t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\left(1-\frac{i}{\hbar}\epsilon V(x,t)\right)\left(\psi(x,t)+\eta\frac{\partial\psi}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\right)d\eta$$

Rewrite as

$$\psi(x,t+\epsilon) = C\left(1-\frac{i}{\hbar}\epsilon V(x,t)\right)(I_1+I_2+I_3)$$

where

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\psi\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\psi \\
I_2 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\eta\frac{\partial\psi}{\partial x}\,d\eta &&= 0 \\
I_3 &= \int_{-\infty}^{\infty} \exp\left(\frac{im\eta^2}{2\hbar\epsilon}\right)\frac{\eta^2}{2}\frac{\partial^2\psi}{\partial x^2}\,d\eta &&= \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{\frac{1}{2}}\frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2}
\end{aligned}$$

Integral $I_1$ is solved by the identity

$$\int_{-\infty}^{\infty}\exp(ay^2)\,dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}}$$

Integral $I_2$ vanishes by the identity

$$\int_{-\infty}^{\infty} y\exp(ay^2)\,dy = 0$$

Integral $I_3$ is solved by the identity

$$\int_{-\infty}^{\infty} y^2\exp(ay^2)\,dy = \left(-\frac{\pi}{a}\right)^{\frac{1}{2}}\left(-\frac{1}{2a}\right)$$

Let

$$C = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{1}{2}}$$

and substitute the solved integrals to obtain

$$\psi(x,t+\epsilon) = \left(1-\frac{i}{\hbar}\epsilon V\right)\left(\psi+\frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2}\right)$$

Expand the product.

$$\psi(x,t+\epsilon) = \psi + \frac{i\hbar\epsilon}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}\epsilon V\psi + \frac{\epsilon^2}{2m}V\frac{\partial^2\psi}{\partial x^2}$$

Expand $\psi(x, t+\epsilon)$ as the power series

$$\psi(x, t+\epsilon) \approx \psi + \epsilon \frac{\partial \psi}{\partial t}$$

to obtain

$$\psi + \epsilon \frac{\partial \psi}{\partial t} = \psi + \frac{i\hbar\epsilon}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} \epsilon V \psi + \frac{\epsilon^2}{2m} V \frac{\partial^2 \psi}{\partial x^2}$$

Cancel $\psi$ and multiply both sides by $i\hbar/\epsilon$.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi + \frac{i\hbar\epsilon}{2m} V \frac{\partial^2 \psi}{\partial x^2}$$

The term on the right vanishes for $\epsilon = 0$ hence

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

See Feynman and Hibbs section 4-1.