Scattering part 2

From the Born approximation we have the scattering amplitude

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}\right) V(\mathbf{x})\,d\mathbf{x} \tag{1}$$

where $\mathbf{p}$ is momentum transfer such that

$$\mathbf{p} = \mathbf{p}_i - \mathbf{p}_f$$

and

$$p^2 = |\mathbf{p}|^2 = 4mE(1-\cos\theta)$$

Let $V(\mathbf{x})$ be a spherically symmetric potential $V(r)$. Show that

$$f(\theta,\phi) = -\frac{2m}{\hbar p}\int_0^\infty \sin\left(\frac{pr}{\hbar}\right) V(r)\,r\,dr$$

In polar coordinates and for potential $V(r)$ we have from (1)

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

Integrate over ϕ (multiply by 2π).

$$f(\theta,\phi) = -\frac{m}{\hbar^2}\int_0^\infty\int_0^\pi \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r)\,r^2\sin\theta\,dr\,d\theta$$

Transform the integral over θ into an integral over u where $u = \cos\theta$ and $du = -\sin\theta\,d\theta$. The minus sign in du is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$f(\theta,\phi) = -\frac{m}{\hbar^2}\int_0^\infty\int_{-1}^1 \exp\left(\frac{ipru}{\hbar}\right) V(r)\,r^2\,dr\,du$$

Solve the integral over u.

$$f(\theta,\phi) = -\frac{m}{\hbar^2}\int_0^\infty \left[\frac{\hbar}{ipr}\exp\left(\frac{ipru}{\hbar}\right)\right]_{u=-1}^{u=1} V(r)\,r^2\,dr$$

Cancel r and evaluate limits.

$$f(\theta,\phi) = -\frac{m}{\hbar p}\int_0^\infty \left[-i\exp\left(\frac{ipr}{\hbar}\right) + i\exp\left(-\frac{ipr}{\hbar}\right)\right] V(r)\,r\,dr$$

Hence

$$f(\theta,\phi) = -\frac{2m}{\hbar p}\int_0^\infty \sin\left(\frac{pr}{\hbar}\right) V(r)\,r\,dr$$

We can use $f(\theta)$ instead of $f(\theta,\phi)$ because nothing depends on ϕ.

$$f(\theta) = -\frac{2m}{\hbar p}\int_0^\infty \sin\left(\frac{pr}{\hbar}\right) V(r)\,r\,dr$$

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