Rutherford scattering 2

Find the scattering cross section for screened Coulomb potential $V(r)$.

$$V(r) = -\frac{Ze^2}{4\varepsilon_0 r} \exp\left(-\frac{r}{a}\right)$$

Start with the Born approximation.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d^3\mathbf{r}$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the screened Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -\frac{Ze^2}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ (multiplies Q by 2π).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r \sin\theta\, dr\, d\theta$$

Transform the integral over θ to an integral over y where $y = \cos\theta$ and $dy = -\sin\theta\, d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_{-1}^{1} \int_0^{\infty} \exp\left(\frac{ipry}{\hbar}\right) \exp\left(-\frac{r}{a}\right) r\, dr\, dy$$

Solve the integral over y (note r in the integrand cancels).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\infty} \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right] \exp\left(-\frac{r}{a}\right)\, dr$$

Solve the integral over r.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - 1/a} \exp\left(\frac{ipr}{\hbar} - \frac{r}{a}\right) + \frac{1}{ip/\hbar + 1/a} \exp\left(-\frac{ipr}{\hbar} - \frac{r}{a}\right)\right]_0^{\infty}$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip} \left[-\frac{1}{ip/\hbar - 1/a} - \frac{1}{ip/\hbar + 1/a}\right] = -\frac{Ze^2}{2\varepsilon_0} \frac{2}{(p/\hbar)^2 + (1/a)^2} \tag{1}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2 = \frac{m^2 Z^2 e^4}{4\pi^2 \epsilon_0^2 \big[p^2 + (\hbar/a)^2\big]^2} \tag{2}$$

Substitute $16\pi^2\epsilon_0^2\alpha^2\hbar^2c^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{\left[p^2 + (\hbar/a)^2\right]^2}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 4mE(1 - \cos\theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{\left[4mE(1 - \cos\theta) + (\hbar/a)^2\right]^2}$$

Cancel m^2 in the numerator.

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2\hbar^2c^2}{\left[4E(1 - \cos\theta) + \frac{1}{m}(\hbar/a)^2\right]^2} \tag{3}$$

Let $a \to \infty$ to obtain the ordinary Rutherford cross section formula

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{4E^2(1 - \cos\theta)^2}$$