Rutherford scattering 1

Find the scattering cross section for Coulomb potential $V(r)$.

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$

Start with the Born approximation.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2, \quad Q = \int \exp\left(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) V(\mathbf{r})\, d\mathbf{r}^3$$

Convert Q to polar coordinates.

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) V(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Substitute the Coulomb potential for $V(r,\theta,\phi)$ and note r^2 becomes r.

$$Q = -\frac{Ze^2}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta\, d\phi$$

Integrate over ϕ (multiplies Q by 2π).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\pi} \int_0^{\infty} \exp\left(\frac{ipr\cos\theta}{\hbar}\right) r \sin\theta\, dr\, d\theta$$

Transform the integral over θ to an integral over y where $y = \cos\theta$ and $dy = -\sin\theta\, d\theta$. The minus sign in dy is canceled by interchanging integration limits $\cos 0 = 1$ and $\cos\pi = -1$.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_{-1}^{1} \int_0^{\infty} \exp\left(\frac{ipry}{\hbar}\right) r\, dr\, dy$$

Solve the integral over y (note r in the integrand cancels).

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\infty} \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar}\right) - \exp\left(-\frac{ipr}{\hbar}\right)\right] dr$$

Solve the integral over r.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \frac{\hbar}{ip} \left[\frac{\hbar}{ip} \exp\left(\frac{ipr}{\hbar}\right) + \frac{\hbar}{ip} \exp\left(-\frac{ipr}{\hbar}\right)\right]_0^{\infty}$$

The first exponential is a problem so go back and multiply the integrand by $\exp(-\epsilon r)$.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \int_0^{\infty} \frac{\hbar}{ip} \left[\exp\left(\frac{ipr}{\hbar} - \epsilon r\right) - \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right)\right] dr$$

Solve the integral.

$$Q = -\frac{Ze^2}{2\varepsilon_0} \frac{\hbar}{ip} \left[\frac{1}{ip/\hbar - \epsilon} \exp\left(\frac{ipr}{\hbar} - \epsilon r\right) + \frac{1}{ip/\hbar + \epsilon} \exp\left(-\frac{ipr}{\hbar} - \epsilon r\right)\right]_0^{\infty}$$

Evaluate the limits.

$$Q = -\frac{Ze^2}{2\varepsilon_0}\frac{\hbar}{ip}\left(-\frac{1}{ip/\hbar - \epsilon} - \frac{1}{ip/\hbar + \epsilon}\right) = -\frac{Ze^2}{2\varepsilon_0}\frac{2}{(p/\hbar)^2 + \epsilon^2} \tag{1}$$

Set $\epsilon = 0$ to obtain

$$Q = -\frac{Ze^2\hbar^2}{\varepsilon_0 p^2}$$

Calculate the cross section.

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |Q|^2 = \frac{m^2 Z^2 e^4}{4\pi^2\varepsilon_0^2 p^4} \tag{2}$$

Substitute $16\pi^2\varepsilon_0^2\alpha^2\hbar^2c^2$ for e^4.

$$\frac{d\sigma}{d\Omega} = \frac{4m^2Z^2\alpha^2\hbar^2c^2}{p^4}$$

Symbol p is momentum transfer $|\mathbf{p}_i| - |\mathbf{p}_f|$ such that

$$p^2 = 4mE(1 - \cos\theta)$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{4E^2(1 - \cos\theta)^2} \tag{3}$$

Noting that

$$4\sin^4\frac{\theta}{2} = (1 - \cos\theta)^2$$

we have the alternative form of (3)

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2\hbar^2c^2}{16E^2\sin^4(\theta/2)}$$

Experimental data

The following data is from Geiger and Marsden's 1913 paper where y is the number of scattering events.

θ	y
150	22.2
135	27.4
120	33.0
105	47.3
75	136
60	320
45	989
37.5	1760
30	5260
22.5	20300
15	105400

Let x be the momentum transfer part of $d\sigma$.

$$x_i = \frac{1}{(1 - \cos\theta_i)^2}$$

The scattering probability for angle θ_i is x_i normalized by $\sum x = 4529$.

$$\Pr(\theta_i) = \frac{x_i}{4529}$$

Predicted values $\hat{y}_i$ are $\Pr(\theta_i)$ times total scattering events $\sum y = 134295$.

$$\hat{y}_i = \Pr(\theta_i) \times 134295$$

The following table shows the predicted values $\hat{y}$.

θ	y	$\hat{y}$
150	22.2	34.1
135	27.4	40.7
120	33.0	52.7
105	47.3	74.9
75	136	216
60	320	474
45	989	1383
37.5	1760	2778
30	5260	6608
22.5	20300	20471
15	105400	102162

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y}_i)^2}{\sum_i (y_i - \bar{y})^2} = 0.999$$

The result indicates that $d\sigma$ explains 99.9% of the variance in the data.