Rotation operator part 2

Let R be the matrix

$$R = \begin{pmatrix} 1 & -\theta/n \\ \theta/n & 1 \end{pmatrix}$$

Show that

$$\lim_{n\to\infty} R^n = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The problem is simplified by diagonalizing R as follows.

Solve for λ in

$$|R - \lambda I| = 0$$

to obtain eigenvalues

$$\lambda = 1 \pm \frac{i\theta}{n}$$

Solve for eigenvectors to obtain

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ -i \end{pmatrix}$$

Hence matrix R is diagonalized as

$$R = QDQ^{-1}$$

where D is the diagonal matrix of eigenvalues

$$D = \begin{pmatrix} 1 + i\theta/n & 0 \\ 0 & 1 - i\theta/n \end{pmatrix}$$

and Q is the matrix of eigenvectors

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}$$

Noting that

$$R^n = \left(QDQ^{-1}\right)^n = QDQ^{-1}QDQ^{-1} \cdots QDQ^{-1} = QD^nQ^{-1}$$

we have

$$\lim_{n\to\infty} R^n = Q \left(\lim_{n\to\infty} D^n\right) Q^{-1}$$

By definition of exponential we have

$$\lim_{n\to\infty} \left(1 \pm \frac{i\theta}{n}\right)^n = e^{\pm i\theta}$$

Hence

$$\lim_{n\to\infty} D^n = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

and

$$\lim_{n\to\infty} R^n = Q \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} Q^{-1} = \frac{1}{2} \begin{pmatrix} e^{i\theta} + e^{-i\theta} & ie^{i\theta} - ie^{-i\theta} \\ -ie^{i\theta} + ie^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

Eigenmath script