Rotation operator part 1

Let $R_z(\alpha)$ be the rotation operator

$$R_z(\alpha) = \exp\left(\frac{i\alpha}{\hbar}L_z\right) = \exp\left(\alpha\frac{\partial}{\partial\phi}\right)$$

Show that

$$R_z(\alpha)Y_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi+\alpha)$$

For spherical harmonic $Y_{lm}(\theta,\phi)$ we have

$$Y_{lm}(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\phi}$$

Noting that $e^{im\phi}$ is the only factor that depends on ϕ

$$\begin{aligned} R_z(\alpha)e^{im\phi} &= \left(1+\alpha\frac{\partial}{\partial\phi}+\frac{\alpha^2}{2!}\frac{\partial^2}{\partial\phi^2}+\frac{\alpha^3}{3!}\frac{\partial^3}{\partial\phi^3}+\cdots\right)e^{im\phi} \\ &= \left(1+im\alpha+\frac{(im\alpha)^2}{2!}+\frac{(im\alpha)^3}{3!}+\cdots\right)e^{im\phi} \\ &= e^{im\alpha}e^{im\phi} \end{aligned}$$

Hence

$$R_z(\alpha)Y_{lm}(\theta,\phi) = Y_{lm}(\theta,\phi)e^{im\alpha} = Y_{lm}(\theta,\phi+\alpha) \tag{1}$$

Let $R_\theta(\alpha)$ be the rotation operator

$$R_\theta(\alpha) = \exp\left(\alpha\frac{\partial}{\partial\theta}\right)$$

Show that

$$R_\theta(\alpha)Y_{lm}(\theta,\phi) = Y_{lm}(\theta+\alpha,\phi) \tag{2}$$

Eigenmath script