Rotating wave approximation

Let $\Psi(\mathbf{r},t)$ be the following wave function for a two state system.

$$\Psi(\mathbf{r},t) = \psi_a(\mathbf{r})c_a(t)\exp(-\tfrac{i}{\hbar}E_a t) + \psi_b(\mathbf{r})c_b(t)\exp(-\tfrac{i}{\hbar}E_b t)$$

Let $\hat{H}(\mathbf{r},t)$ be the Hamiltonian

$$\hat{H}(\mathbf{r},t) = \hat{H}_0(\mathbf{r}) + \hat{H}_1(\mathbf{r},t)$$

where

$$\hat{H}_0\psi_a = E_a\psi_a, \quad \hat{H}_0\psi_b = E_b\psi_b, \quad \hat{H}_0\Psi = (E_a + E_b)\Psi$$

From the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

we obtain the differential equations

$$\frac{d}{dt}c_a(t) = -\frac{i}{\hbar}\langle\psi_a|\hat{H}_1|\psi_a\rangle c_a(t) - \frac{i}{\hbar}\langle\psi_a|\hat{H}_1|\psi_b\rangle \exp(-i\omega_0 t)c_b(t)$$
$$\frac{d}{dt}c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{H}_1|\psi_b\rangle c_b(t) - \frac{i}{\hbar}\langle\psi_b|\hat{H}_1|\psi_a\rangle \exp(i\omega_0 t)c_a(t)$$

where

$$\omega_0 = \frac{E_b - E_a}{\hbar}$$

Typically the diagonal elements vanish

$$\langle\psi_a|\hat{H}_1|\psi_a\rangle = \langle\psi_b|\hat{H}_1|\psi_b\rangle = 0$$

and the differential equations become

$$\frac{d}{dt}c_a(t) = -\frac{i}{\hbar}\langle\psi_a|\hat{H}_1|\psi_b\rangle \exp(-i\omega_0 t)c_b(t) \tag{1}$$
$$\frac{d}{dt}c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{H}_1|\psi_a\rangle \exp(i\omega_0 t)c_a(t) \tag{2}$$

Let $\hat{H}_1(\mathbf{r},t)$ be the perturbation

$$\hat{H}_1(\mathbf{r},t) = \hat{V}(\mathbf{r})\cos(\omega t)$$

Then

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \langle\psi_a|\hat{V}|\psi_b\rangle\left[\tfrac{1}{2}\exp(i\omega t) + \tfrac{1}{2}\exp(-i\omega t)\right]$$

The rotating wave approximation discards the second term and asserts

$$\langle\psi_a|\hat{H}_1|\psi_b\rangle = \tfrac{1}{2}\langle\psi_a|\hat{V}|\psi_b\rangle \exp(i\omega t) \tag{3}$$

Substitute equation (3) into (1) and (2) to obtain

$$\frac{d}{dt}c_a(t) = -\frac{i}{2\hbar}\langle\psi_a|\hat{V}|\psi_b\rangle \exp\bigl(i(\omega-\omega_0)t\bigr)c_b(t) \tag{4}$$

$$\frac{d}{dt}c_b(t) = -\frac{i}{2\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle \exp\bigl(i(\omega_0-\omega)t\bigr)c_a(t) \tag{5}$$

Use Laplace transforms to solve for $c_b(t)$ with initial conditions $c_a(0) = 1$ and $c_b(0) = 0$.

$$c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle \frac{\sin(\omega_r t)}{2\omega_r} \exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right) \tag{6}$$

Symbol ω_r is the Rabi flopping frequency

$$\omega_r = \frac{1}{2}\sqrt{(\omega_0-\omega)^2 + \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2/\hbar^2}$$

Use equation (2) and the solution for $c_b(t)$ to solve for $c_a(t)$.

$$c_a(t) = \left[\cos(\omega_r t) + i\left(\frac{\omega_0-\omega}{2\omega_r}\right)\sin(\omega_r t)\right]\exp\left(-\tfrac{i}{2}(\omega_0-\omega)t\right)$$

Rewrite ω_r as

$$\omega_r = \frac{1}{2\hbar}\sqrt{\hbar^2(\omega_0-\omega)^2 + \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2}$$

and note that for

$$\hbar^2(\omega_0-\omega)^2 \gg \left|\langle\psi_a|\hat{V}|\psi_b\rangle\right|^2$$

we have

$$\omega_r \approx \tfrac{1}{2}|\omega_0-\omega| \tag{7}$$

Substitute (7) into (6) to obtain

$$c_b(t) = -\frac{i}{\hbar}\langle\psi_b|\hat{V}|\psi_a\rangle \frac{\sin\left(\frac{1}{2}|\omega_0-\omega|t\right)}{|\omega_0-\omega|}\exp\left(\tfrac{i}{2}(\omega_0-\omega)t\right)$$

This is equivalent to $c_b(t)$ obtained from first order perturbation expansion.