Quantum harmonic oscillator

Anything quadratic is called harmonic. —A. Zee

A harmonic oscillator is anything with potential energy proportional to displacement squared.

$$V(x) \propto x^2$$

For a quantum harmonic oscillator

$$V(x) = \frac{m\omega^2 x^2}{2}$$

Hence the hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2}, \quad \hat{p} = -i\hbar\frac{d}{dx}$$

We seek to solve the eigenvalue equation

$$\hat{H}\psi_n = E_n\psi_n$$

The solution is

$$\psi_n(x) = C_n \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(x\sqrt{m\omega/\hbar}\right), \quad n = 0, 1, 2, \ldots$$

C_n is the normalization constant

$$C_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}$$

H_n is the nth hermite polynomial

$$H_n(y) = (-1)^2 \exp\left(y^2\right) \frac{d^n}{dy^n} \exp\left(-y^2\right)$$

The eigenvalues are

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

The ladder operators are

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i\hat{p}}{m\omega}\right)$$

Operator $\hat{a}$ lowers ψ_n.

$$\hat{a}\psi_n = \sqrt{n}\psi_{n-1}$$

Operator $\hat{a}^\dagger$ raises ψ_n.

$$\hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$$

This is how ψ_n can be obtained from ψ_0.

$$\psi_n = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\psi_0$$

The number operator is the result of lowering then raising.

$$\hat{N} = \hat{a}^\dagger\hat{a}, \quad \hat{N}\psi_n = n\psi_n$$

Exercises

1. Verify ψ_n and E_n.

2. Verify ladder operators.

3. Let

$$\Psi(x) = \frac{\psi_2(x) + \psi_3(x)}{\sqrt{2}}$$

Verify that

$$\Pr(x \geq 0) = \int_0^\infty \Psi^* \Psi \, dx \approx 0.85$$

4. Let

$$m = 6.64 \times 10^{-27} \,\text{kilogram}, \quad V(10^{-6}\,\text{meter}) = 1\,\text{electronvolt}$$

Verify that

$$\omega = \sqrt{\frac{2V(x)}{mx^2}} = 6.95 \times 10^9 \,\text{second}^{-1}$$

For $\Psi = (\psi_2 + \psi_3)/\sqrt{2}$ verify that

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x \Psi^* \Psi \, dx = 1.85 \times 10^{-9}\,\text{meter} \\
\langle E \rangle &= \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi \, dx = 1.37 \times 10^{-5}\,\text{electronvolt} = \frac{E_2 + E_3}{2}
\end{aligned}$$