Particle in a box

The "box" is a potential $V(x)$ such that

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

We seek $\psi(x,t)$ that solves the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$

For $V(x) = 0$ we have the solution

$$\psi(x,t) = [A\sin(kx) + B\cos(kx)]\exp(-i\omega t)$$

where

$$\omega = \frac{k^2\hbar}{2m}$$

By the boundary condition $\psi(0,t) = 0$ we have

$$B = 0$$

By the boundary condition $\psi(a,t) = 0$ we have for integer n

$$kx = \frac{n\pi x}{a}$$

We seek A that normalizes probability density $|\psi|^2$.

$$\int_0^a |\psi|^2\,dx = \frac{a}{2}|A|^2 = 1$$

Hence

$$|A| = \sqrt{\frac{2}{a}}$$

Let E_n be particle energy $\hbar\omega$.

$$E_n = \hbar\omega = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Put everything together and write

$$\psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{iE_n t}{\hbar}\right)$$

Check dimensions.

$$\omega = \frac{k^2\hbar}{2m} = \frac{[\mathrm{m}^{-2}]\,[\mathrm{J\,s}]}{[\mathrm{kg}]} = [\mathrm{s}^{-1}]$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{[\mathrm{J}^2\,\mathrm{s}^2]}{[\mathrm{kg}]\,[\mathrm{m}^2]} = [\mathrm{J}]$$

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