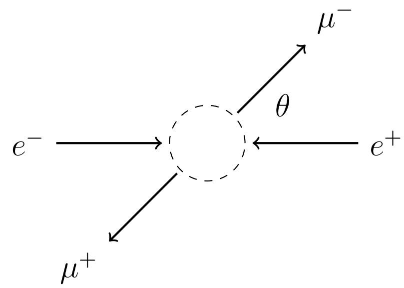
Muon pair production

Muon pair production is the interaction $e^- + e^+ \to \mu^- + \mu^+$.



Define the following momentum vectors and spinors. Symbol E is beam energy. Symbol p is electron momentum $p = \sqrt{E^2 - m^2}$ where m is electron mass 0.51 MeV. Symbol ρ is muon mementum $\rho = \sqrt{E^2 - M^2}$ where M is muon mass 106 MeV. Polar angle θ is the observed scattering angle. Azimuth angle ϕ cancels out in scattering calculations.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}$$
inbound e^-

$$u_{11} = \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}$$
inbound e^-
spin up

$$u_{12} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}$$
inbound e^-
spin down

$$p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}$$
inbound e^+

$$v_{21} = \begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}$$
inbound e^+
spin up

$$v_{22} = \begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}$$
inbound e^+
spin down

$$p_3 = \begin{pmatrix} E \\ \rho\sin\theta\cos\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\theta \end{pmatrix}$$
outbound μ^-

$$u_{31} = \begin{pmatrix} E+M \\ 0 \\ p_3^z \\ p_3^x + ip_3^y \end{pmatrix}$$
outbound μ^-
spin up

$$u_{32} = \begin{pmatrix} 0 \\ E+M \\ p_3^x - ip_3^y \\ -p_3^z \end{pmatrix}$$
outbound μ^-
spin down

$$p_4 = \begin{pmatrix} E \\ -\rho\sin\theta\cos\phi \\ -\rho\sin\theta\sin\phi \\ -\rho\cos\theta \end{pmatrix}$$
outbound μ^+

$$v_{41} = \begin{pmatrix} p_4^z \\ p_4^x + ip_4^y \\ E+M \\ 0 \end{pmatrix}$$
outbound μ^+
spin up

$$v_{42} = \begin{pmatrix} p_4^x - ip_4^y \\ -p_4^z \\ 0 \\ E+M \end{pmatrix}$$
outbound μ^+
spin down

The spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2(E+M)^2$ will be used.

This is the probability density for spin state *abcd*. The formula is derived from Feynman diagrams for muon pair production.

$$\left|\mathcal{M}_{abcd}\right|^2 = \frac{e^4}{Ns^2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2$$

Symbol e is electron charge and

$$s = (p_1 + p_2)^2 = 4E^2$$

The expected probability density $\langle|\mathcal{M}|^2\rangle$ is computed by summing $|\mathcal{M}_{abcd}|^2$ over all spin states and dividing by the number of inbound states. There are four inbound states.

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|\mathcal{M}_{abcd}\right|^2 \\ &= \frac{e^4}{4Ns^2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}\left|(\bar{u}_{3c}\gamma_\mu v_{4d})(\bar{v}_{2b}\gamma^\mu u_{1a})\right|^2\end{aligned}$$

The Casimir trick uses matrix arithmetic to compute $\langle|\mathcal{M}|^2\rangle$.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{4s^2}\operatorname{Tr}\Big((\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\Big)\operatorname{Tr}\Big((\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\Big)$$

The result is

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right)$$

For high energy experiments $E \gg M$ a useful approximation is

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta\right)$$

Cross section

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\langle|\mathcal{M}|^2\rangle}{4(4\pi\varepsilon_0)^2 s}, \quad s = (p_1 + p_2)^2 = 4E^2$$

For high energy experiments we have

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta\right)$$

Hence for high energy experiments

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi\varepsilon_0)^2 s}\left(1 + \cos^2\theta\right)$$

Noting that

$$e^2 = 4\pi\varepsilon_0\alpha\hbar c$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar c)^2}{4s}\left(1+\cos^2\theta\right)$$

Noting that

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

we also have

$$d\sigma = \frac{\alpha^2(\hbar c)^2}{4s}\left(1+\cos^2\theta\right)\sin\theta\, d\theta\, d\phi$$

Let $S(\theta_1,\theta_2)$ be the following surface integral of $d\sigma$.

$$S(\theta_1,\theta_2) = \int_0^{2\pi}\int_{\theta_1}^{\theta_2} d\sigma$$

The solution is

$$S(\theta_1,\theta_2) = \frac{2\pi\alpha^2(\hbar c)^2}{4s}\big(I(\theta_2)-I(\theta_1)\big)$$

where

$$I(\theta) = -\frac{\cos^3\theta}{3} - \cos\theta$$

The cumulative distribution function is

$$F(\theta) = \frac{S(0,\theta)}{S(0,\pi)} = \frac{I(\theta)-I(0)}{I(\pi)-I(0)} = -\frac{\cos^3\theta}{8} - \frac{3\cos\theta}{8} + \frac{1}{2}, \quad 0\le\theta\le\pi$$

The probability of observing scattering events in the interval θ_1 to θ_2 is

$$P(\theta_1\le\theta\le\theta_2) = F(\theta_2) - F(\theta_1)$$

Let N be the total number of scattering events from an experiment. Then the number of scattering events in the interval θ_1 to θ_2 is predicted to be

$$NP(\theta_1\le\theta\le\theta_2)$$

The probability density function is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{3}{8}\left(1+\cos^2\theta\right)\sin\theta$$

Data from SLAC PEP experiment

See www.hepdata.net/record/ins216031, Table 1, $s = (29.0\,\mathrm{GeV})^2$.

x	y
-0.925	67.08
-0.85	58.67
-0.75	54.66
-0.65	51.72
-0.55	43.70
-0.45	41.12
-0.35	39.71
-0.25	35.34
-0.15	33.35
-0.05	34.69
0.05	34.05
0.15	34.48
0.25	34.66
0.35	35.23
0.45	35.60
0.55	40.13
0.65	42.56
0.75	46.37
0.85	49.28
0.925	55.70

Data x and y have the following relationship with the differential cross section formula.

$$x = \cos\theta, \quad y = s\frac{d\sigma}{d\cos\theta} = 2\pi s\frac{d\sigma}{d\Omega}$$

The cross section formula is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \times (\hbar c)^2$$

To compute predicted values $\hat{y}$, multiply by 10^{37} to convert square meters to nanobarns.

$$\hat{y} = 2\pi s\frac{d\sigma}{d\Omega} = \frac{\pi\alpha^2}{2}\left(1+x^2\right) \times (\hbar c)^2 \times 10^{37}$$

The following table shows predicted values $\hat{y}$.

x	y	$\hat{y}$
-0.925	67.08	60.44
-0.85	58.67	56.10
-0.75	54.66	50.89
-0.65	51.72	46.33
-0.55	43.70	42.42
-0.45	41.12	39.17
-0.35	39.71	36.56
-0.25	35.34	34.61
-0.15	33.35	33.30
-0.05	34.69	32.65
0.05	34.05	32.65
0.15	34.48	33.30
0.25	34.66	34.61
0.35	35.23	36.56
0.45	35.60	39.17
0.55	40.13	42.42
0.65	42.56	46.33
0.75	46.37	50.89
0.85	49.28	56.10
0.925	55.70	60.44

The coefficient of determination R^2 measures how well predicted values fit the data.

$$R^2 = 1 - \frac{\sum(y - \hat{y})^2}{\sum(y - \bar{y})^2} = 0.87$$

The result indicates that the model $d\sigma$ explains 87% of the variance in the data.

Electroweak model

The following differential cross section formula from electroweak theory results in a better fit to the data.[1]

$$\frac{d\sigma}{d\Omega} = F(s)\big(1 + \cos^2\theta\big) + G(s)\cos\theta$$

where

$$F(s) = \frac{\alpha^2}{4s}\left(1 + \frac{g_V^2}{\sqrt{2}\pi}\left(\frac{m_Z^2}{s - m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{(g_A^2 + g_V^2)^2}{8\pi^2}\left(\frac{m_Z^2}{s - m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

$$G(s) = \frac{\alpha^2}{4s}\left(\frac{\sqrt{2}g_A^2}{\pi}\left(\frac{m_Z^2}{s - m_Z^2}\right)\left(\frac{sG}{\alpha}\right) + \frac{g_A^2 g_V^2}{\pi^2}\left(\frac{m_Z^2}{s - m_Z^2}\right)^2\left(\frac{sG}{\alpha}\right)^2\right)$$

[1] F. Mandl and G. Shaw, *Quantum Field Theory Revised Edition,* 316.

and

$$\begin{aligned}
g_A &= -0.5 \\
g_V &= -0.0348 \\
m_Z &= 91.17\,\text{GeV} \\
G &= 1.166 \times 10^{-5}\,\text{GeV}^{-2}
\end{aligned}$$

The corresponding formula for $\hat{y}$ is

$$\hat{y} = 2\pi \left[F(s)(1 + x^2) + G(s)x \right] \times (\hbar c)^2 \times 10^{37}$$

where $\sqrt{s} = 29\,\text{GeV}$ is the center of mass collision energy. Here are the predicted values $\hat{y}$ based on the above formula.

x	y	$\hat{y}$
−0.925	67.08	65.59
−0.85	58.67	60.84
−0.75	54.66	55.07
−0.65	51.72	49.96
−0.55	43.70	45.49
−0.45	41.12	41.69
−0.35	39.71	38.53
−0.25	35.34	36.02
−0.15	33.35	34.17
−0.05	34.69	32.97
0.05	34.05	32.42
0.15	34.48	32.53
0.25	34.66	33.28
0.35	35.23	34.69
0.45	35.60	36.75
0.55	40.13	39.47
0.65	42.56	42.83
0.75	46.37	46.85
0.85	49.28	51.52
0.925	55.70	55.45

The coefficient of determination R^2 is

$$R^2 = 1 - \frac{\sum (y - \hat{y})^2}{\sum (y - \bar{y})^2} = 0.98$$

The result indicates that electroweak theory explains 98% of the variance in the data.

Notes

Here are a few notes about how the demo script works.

In component notation, traces are sums over a repeated index, in this case α.

$$\operatorname{Tr}\Big((\not p_3 + M)\gamma^\mu(\not p_4 - M)\gamma^\nu\Big) = (\not p_3 + M)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_4 - M)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha$$
$$\operatorname{Tr}\Big((\not p_2 - m)\gamma_\mu(\not p_1 + m)\gamma_\nu\Big) = (\not p_2 - m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_1 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha$$

To convert the above formulas to Eigenmath code, the γ tensors need to be transposed so that repeated indices are adjacent to each other. Also, multiply γ^μ by the metric tensor to lower the index.

$\gamma^{\beta\mu}{}_\rho$	$\rightarrow$	`gammaT = transpose(gamma)`
$\gamma^\beta{}_{\mu\rho}$	$\rightarrow$	`gammaL = transpose(dot(gmunu,gamma))`

Define the following 4×4 matrices.

$(\not p_1 + m)$	$\rightarrow$	`X1 = pslash1 + m I`
$(\not p_2 - m)$	$\rightarrow$	`X2 = pslash2 - m I`
$(\not p_3 + M)$	$\rightarrow$	`X3 = pslash3 + M I`
$(\not p_4 - M)$	$\rightarrow$	`X4 = pslash4 - M I`

Then

$(\not p_3 + M)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not p_4 - M)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\alpha$	$\rightarrow$	`T1 = contract(dot(X3,gammaT,X4,gammaT),1,4)`
$(\not p_2 - m)^\alpha{}_\beta \gamma_\mu{}^\beta{}_\rho (\not p_1 + m)^\rho{}_\sigma \gamma_\nu{}^\sigma{}_\alpha$	$\rightarrow$	`T2 = contract(dot(X2,gammaL,X1,gammaL),1,4)`

Next, multiply matrices and sum over repeated indices. The dot function sums over ν then the contract function sums over μ. The transpose makes the ν indices adjacent as required by the dot function.

$\operatorname{Tr}(\cdots\gamma^\mu\cdots\gamma^\nu)\operatorname{Tr}(\cdots\gamma_\mu\cdots\gamma_\nu)$	$\rightarrow$	`contract(dot(T1,transpose(T2)))`