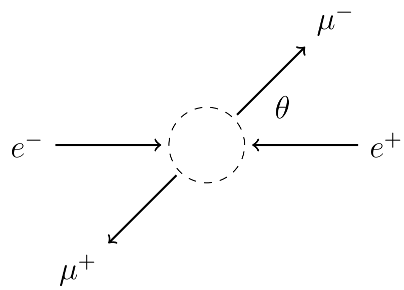
Muon pair production 1

Muon pair production is the process $e^- + e^+ \to \mu^- + \mu^+$.



For the following momentum vectors we have $p = \sqrt{E^2 - m^2}$ and $\rho = \sqrt{E^2 - M^2}$ where m is electron mass and M is muon mass.

$$p_1 = \underset{e^- \longrightarrow}{\begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}} \qquad p_2 = \underset{\longleftarrow e^+}{\begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}} \qquad p_3 = \underset{\mu^- \nearrow}{\begin{pmatrix} E \\ \rho \sin\theta \cos\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\theta \end{pmatrix}} \qquad p_4 = \underset{\swarrow \mu^+}{\begin{pmatrix} E \\ -\rho \sin\theta \cos\phi \\ -\rho \sin\theta \sin\phi \\ -\rho \cos\theta \end{pmatrix}}$$

Spinors for p_1.

$$u_{11} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}} \qquad u_{12} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}}$$

Spinors for p_2.

$$v_{21} = \frac{1}{\sqrt{E+m}} \underset{\text{spin up}}{\begin{pmatrix} -p \\ 0 \\ E+m \\ 0 \end{pmatrix}} \qquad v_{22} = \frac{1}{\sqrt{E+m}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ p \\ 0 \\ E+m \end{pmatrix}}$$

Spinors for p_3.

$$u_{31} = \frac{1}{\sqrt{E+M}} \underset{\text{spin up}}{\begin{pmatrix} E+M \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix}} \qquad u_{32} = \frac{1}{\sqrt{E+M}} \underset{\text{spin down}}{\begin{pmatrix} 0 \\ E+M \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix}}$$

Spinors for p_4.

$$v_{41} = \frac{1}{\sqrt{E+M}} \underset{\text{spin up}}{\begin{pmatrix} p_{4z} \\ p_{4x} + ip_{4y} \\ E+M \\ 0 \end{pmatrix}} \qquad v_{42} = \frac{1}{\sqrt{E+M}} \underset{\text{spin down}}{\begin{pmatrix} p_{4x} - ip_{4y} \\ -p_{4z} \\ 0 \\ E+M \end{pmatrix}}$$

The scattering amplitude $\mathcal{M}_{abcd}$ for spin state $abcd$ is

$$\mathcal{M}_{abcd} = \frac{e^2}{4E^2}(\bar{u}_{3c}\gamma^\mu v_{4d})(\bar{v}_{2b}\gamma_\mu u_{1a})$$

In component form

$$\mathcal{M}_{abcd} = \frac{e^2}{4E^2}\left[(\bar{u}_{3c})_\alpha \gamma^{\mu\alpha}{}_\beta (v_{4d})^\beta\right]\left[(\bar{v}_{2b})_\rho \gamma_\mu{}^\rho{}_\sigma (u_{1a})^\sigma\right]$$

Scattering density $|\mathcal{M}_{abcd}|^2$ is the squared magnitude of scattering amplitude $\mathcal{M}_{abcd}$.

$$|\mathcal{M}_{abcd}|^2 = \mathcal{M}^*_{abcd}\mathcal{M}_{abcd}$$

Average scattering density $\langle|\mathcal{M}|^2\rangle$ is the sum over all scattering densities divided by the number of inbound states.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{abcd}|\mathcal{M}_{abcd}|^2$$

The Casimir trick uses matrix arithmetic to sum over densities.

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{64E^4}\operatorname{Tr}\left[(\not{p}_3 + M)\gamma^\mu(\not{p}_4 - M)\gamma^\nu\right]\operatorname{Tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right]$$

The result is

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta + \frac{m^2 + M^2}{E^2}\sin^2\theta + \frac{m^2M^2}{E^4}\cos^2\theta\right)$$

For high energy experiments where $E \gg M$ we use the following approximation.

$$\langle|\mathcal{M}|^2\rangle = e^4\left(1 + \cos^2\theta\right)$$

The scattering cross section is

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha\hbar c)^2}{16E^2}\frac{\langle|\mathcal{M}|^2\rangle}{e^4}$$

Hence for $E \gg M$

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha\hbar c)^2}{16E^2}\left(1 + \cos^2\theta\right)$$

In terms of Mandelstam variable s where

$$s = (p_1 + p_2)^2 = 4E^2$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha\hbar c)^2}{4s}\left(1 + \cos^2\theta\right)$$