Muon decay

The following muon data is from Particle Data Group.

From "V minus A" theory we have the following formula for muon lifetime τ.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5}$$

Symbol G_F is Fermi coupling constant, m_μ is muon mass.

From NIST we have

$$G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}$$
$$m_\mu = 1.883531627 \times 10^{-28} \text{ kilogram}$$
$$h = 6.62607015 \times 10^{-34} \text{ joule second (exact)}$$
$$c = 299792458 \text{ meter second}^{-1} \text{ (exact)}$$
$$1\,\text{eV} = 1.602176634 \times 10^{-19} \text{ joule (exact)}$$

Hence

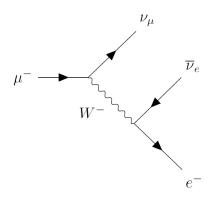$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5} = 2.18735 \times 10^{-6} \text{ second}$$

The result is a bit smaller than the observed value from Particle Data Group.

$$\frac{\tau}{\text{observed value}} = \frac{2.18735 \times 10^{-6} \text{ second}}{2.19698 \times 10^{-6} \text{ second}} = 0.9956$$

As the following diagram shows, a muon decays into a muon neutrino, an electron anti-neutrino, and an electron.

$$\nu_\mu$$
$$\overline{\nu}_e$$
$$\mu^-$$
$$W^-$$
$$e^-$$

Particle	Symbol	Momentum	Spinor (up)	Spinor (down)
Muon	μ^-	p_1	u_{11}	u_{12}
Muon neutrino	ν_μ	p_2	u_{21}	u_{22}
Electron anti-neutrino	$\bar{\nu}_e$	p_3	v_{31}	v_{32}
Electron	e^-	p_4	u_{41}	u_{42}

We will use the following momentum vectors.

$$p_1 = \begin{pmatrix} E_1 \\ p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix} \qquad p_2 = \begin{pmatrix} E_2 \\ p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix} \qquad p_3 = \begin{pmatrix} E_3 \\ p_{3x} \\ p_{3y} \\ p_{3z} \end{pmatrix} \qquad p_4 = \begin{pmatrix} E_4 \\ p_{4x} \\ p_{4y} \\ p_{4z} \end{pmatrix}$$

$$\mu^- \qquad\qquad \nu_\mu \qquad\qquad \bar{\nu}_e \qquad\qquad e^-$$

And we will use the following unnormalized spinors.

$$u_{11} = \begin{pmatrix} E_1 + m_\mu \\ 0 \\ p_{1z} \\ p_{1x} + ip_{1y} \end{pmatrix} \quad u_{21} = \begin{pmatrix} E_2 + m_2 \\ 0 \\ p_{2z} \\ p_{2x} + ip_{2y} \end{pmatrix} \quad v_{31} = \begin{pmatrix} p_{3z} \\ p_{3x} + ip_{3y} \\ E_3 + m_3 \\ 0 \end{pmatrix} \quad u_{41} = \begin{pmatrix} E_4 + m_4 \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}$$

$$\mu^- \text{ spin up} \qquad\qquad \nu_\mu \text{ spin up} \qquad\qquad \bar{\nu}_e \text{ spin up} \qquad\qquad e^- \text{ spin up}$$

$$u_{12} = \begin{pmatrix} 0 \\ E_1 + m_\mu \\ p_{1x} - ip_{1y} \\ -p_{1z} \end{pmatrix} \quad u_{22} = \begin{pmatrix} 0 \\ E_2 + m_2 \\ p_{2x} - ip_{2y} \\ -p_{2z} \end{pmatrix} \quad v_{32} = \begin{pmatrix} p_{3x} - ip_{3y} \\ -p_{3z} \\ 0 \\ E_3 + m_3 \end{pmatrix} \quad u_{42} = \begin{pmatrix} 0 \\ E_4 + m_4 \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}$$

$$\mu^- \text{ spin down} \qquad\qquad \nu_\mu \text{ spin down} \qquad\qquad \bar{\nu}_e \text{ spin down} \qquad\qquad e^- \text{ spin down}$$

Symbol E_n is total energy of particle n.

$$E_1 = \sqrt{(p_{1x})^2 + (p_{1y})^2 + (p_{1z})^2 + m_\mu^2}$$

$$E_2 = \sqrt{(p_{2x})^2 + (p_{2y})^2 + (p_{2z})^2 + m_2^2}$$

$$E_3 = \sqrt{(p_{3x})^2 + (p_{3y})^2 + (p_{3z})^2 + m_3^2}$$

$$E_4 = \sqrt{(p_{4x})^2 + (p_{4y})^2 + (p_{4z})^2 + m_4^2}$$

This is the probability amplitude for spin state $abcd$. The formula is derived from Feynman diagrams for muon decay.

$$\mathcal{M}_{abcd} = \frac{G_F}{\sqrt{2N}} \underbrace{\left(\bar{u}_{4d}\,\gamma^\mu(1-\gamma^5)\,v_{3c}\right)}_{e^- \qquad \bar{\nu}_e} \underbrace{\left(\bar{u}_{2b}\,\gamma_\mu(1-\gamma^5)\,u_{1a}\right)}_{\nu_\mu \qquad \mu^-}$$

Symbol N is the following spinor normalization constant.

$$N = (E_1 + m_\mu)(E_2 + m_2)(E_3 + m_3)(E_4 + m_4)$$

Recall that the magnitude squared of an amplitude is a probability density.

$$|\mathcal{M}_{abcd}|^2 = \mathcal{M}^*_{abcd}\mathcal{M}_{abcd}$$

In a typical muon decay experiment the spins are not observed. Consequently, the experimental result is an average of spin states. The average is computed by summing over all spin states and dividing by the number of initial spin states. The muon has two spin states hence the divisor is two.

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{2}\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}|\mathcal{M}_{abcd}|^2$$

The result is a simple formula.

$$\langle|\mathcal{M}|^2\rangle = 64G_F^2(p_1 \cdot p_3)(p_2 \cdot p_4) \tag{1}$$

In component notation we have

$$\langle|\mathcal{M}|^2\rangle = 64G_F^2(p_1^\mu\,g_{\mu\nu}\,p_3^\nu)(p_2^\rho\,g_{\rho\sigma}\,p_4^\sigma)$$

where

$$g_{\mu\nu} = g_{\rho\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Muon decay rate Γ is essentially an expectation value (weighted average) for all possible values of particle momentum. In the muon rest frame we have $p_1 = (m_\mu, 0, 0, 0)$ and by Fermi's golden rule

$$\Gamma = \frac{1}{512\pi^5 m_\mu} \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} \langle|\mathcal{M}|^2\rangle\,\delta(p_1 - p_2 - p_3 - p_4)\,\frac{d^3p_2}{E_2}\,\frac{d^3p_3}{E_3}\,\frac{d^3p_4}{E_4}$$

Altogether there are nine integrals, three for each of p_2, p_3, and p_4. The delta function restricts the integration space to values that conserve energy and momentum.

It can be shown that

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Muon lifetime τ is the inverse of decay rate.

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Change natural units to h and c.

$$\tau = \frac{96\pi^2 h}{G_F^2 \left(m_\mu c^2\right)^5}$$

Probability density $\langle |\mathcal{M}|^2 \rangle$ can also be computed using the following Casimir trick.

$$\langle |\mathcal{M}|^2 \rangle = \frac{G_F^2}{4} \operatorname{Tr}\left(\slashed{p}_4 \gamma^\mu (1 - \gamma^5) \slashed{p}_3 \gamma^\nu (1 - \gamma^5)\right) \operatorname{Tr}\left(\slashed{p}_2 \gamma_\mu (1 - \gamma^5) \slashed{p}_1 \gamma_\nu (1 - \gamma^5)\right)$$

The slashed symbols are 4×4 matrices computed as

$$\slashed{p} = p \cdot \gamma = p^0 \gamma^0 - p^1 \gamma^1 - p^2 \gamma^2 - p^3 \gamma^3$$

For example,

$$\slashed{p}_1 = \begin{pmatrix} E_1 & 0 & -p_{1z} & -p_{1x} + ip_{1y} \\ 0 & E_1 & -p_{1x} - ip_{1y} & p_{1z} \\ p_{1z} & p_{1x} - ip_{1y} & -E_1 & 0 \\ p_{1x} + ip_{1y} & -p_{1z} & 0 & -E_1 \end{pmatrix}$$