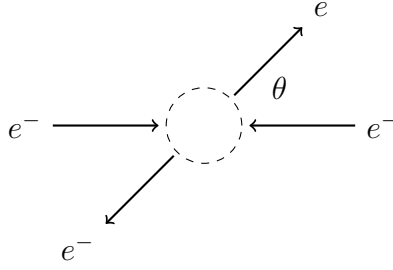


Moller scattering 1

Moller scattering is the process $e^- + e^- \rightarrow e^- + e^-$.



Momentum vectors where $E = \sqrt{p^2 + m^2}$.

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ p \end{pmatrix}_{e^- \rightarrow} \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -p \end{pmatrix}_{\leftarrow e^-} \quad p_3 = \begin{pmatrix} E \\ p \sin \theta \cos \phi \\ p \sin \theta \sin \phi \\ p \cos \theta \end{pmatrix}_{\nearrow e^-} \quad p_4 = \begin{pmatrix} E \\ -p \sin \theta \cos \phi \\ -p \sin \theta \sin \phi \\ -p \cos \theta \end{pmatrix}_{e^- \nwarrow}$$

Spinors for p_1 .

$$u_{11} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p \\ 0 \end{pmatrix}_{\text{spin up}} \quad u_{12} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p \end{pmatrix}_{\text{spin down}}$$

Spinors for p_2 .

$$u_{21} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ -p \\ 0 \end{pmatrix}_{\text{spin up}} \quad u_{22} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ 0 \\ p \end{pmatrix}_{\text{spin down}}$$

Spinors for p_3 .

$$u_{31} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{3z} \\ p_{3x} + ip_{3y} \end{pmatrix}_{\text{spin up}} \quad u_{32} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{3x} - ip_{3y} \\ -p_{3z} \end{pmatrix}_{\text{spin down}}$$

Spinors for p_4 .

$$u_{41} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} E+m \\ 0 \\ p_{4z} \\ p_{4x} + ip_{4y} \end{pmatrix}_{\text{spin up}} \quad u_{42} = \frac{1}{\sqrt{E+m}} \begin{pmatrix} 0 \\ E+m \\ p_{4x} - ip_{4y} \\ -p_{4z} \end{pmatrix}_{\text{spin down}}$$

The scattering amplitude $\mathcal{M}$ is

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

where $\mathcal{M}_1$ is the $1 \rightarrow 3, 2 \rightarrow 4$ amplitude

$$\mathcal{M}_1 = \frac{e^2}{t} (\bar{u}_{4d} \gamma^\mu u_{2b}) (\bar{u}_{3c} \gamma_\mu u_{1a})$$

and $\mathcal{M}_2$ is the $1 \rightarrow 4, 2 \rightarrow 3$ amplitude

$$\mathcal{M}_2 = \frac{e^2}{u} (\bar{u}_{3c} \gamma^\nu u_{2b}) (\bar{u}_{4d} \gamma_\nu u_{1a})$$

The expected density $\langle |\mathcal{M}|^2 \rangle$ is the average density.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} |\mathcal{M}|^2 = \frac{1}{4} \sum_{abcd} (\mathcal{M}_1 - \mathcal{M}_2)(\mathcal{M}_1 - \mathcal{M}_2)^*$$

Expand the summand.

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{abcd} (\mathcal{M}_1 \mathcal{M}_1^* - \mathcal{M}_1 \mathcal{M}_2^* - \mathcal{M}_2 \mathcal{M}_1^* + \mathcal{M}_2 \mathcal{M}_2^*)$$

The summation results are

$$\begin{aligned} \sum_{abcd} \mathcal{M}_1 \mathcal{M}_1^* &= \frac{e^4}{t^2} (8s^2 + 8u^2 - 64m^2s - 64m^2u + 192m^4) \\ \sum_{abcd} \mathcal{M}_1 \mathcal{M}_2^* &= \frac{e^4}{tu} (-8s^2 + 64m^2s - 96m^4) \\ \sum_{abcd} \mathcal{M}_2 \mathcal{M}_2^* &= \frac{e^4}{u^2} (8s^2 + 8t^2 - 64m^2s - 64m^2t + 192m^4) \end{aligned}$$

where s , t , and u are Mandelstam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2 \end{aligned}$$

For clarity define

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 - 64m^2s - 64m^2u + 192m^4 \\ f_{12} &= -8s^2 + 64m^2s - 96m^4 \\ f_{22} &= 8s^2 + 8t^2 - 64m^2s - 64m^2t + 192m^4 \end{aligned}$$

and write $\langle |\mathcal{M}|^2 \rangle$ more compactly as

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4} \left(\frac{f_{11}}{t^2} - \frac{2f_{12}}{tu} + \frac{f_{22}}{u^2} \right)$$

For $E \gg m$ we use the approximation $m = 0$ and obtain

$$\begin{aligned} f_{11} &= 8s^2 + 8u^2 \\ f_{12} &= -8s^2 \\ f_{22} &= 8s^2 + 8t^2 \end{aligned}$$

Then $\langle |\mathcal{M}|^2 \rangle$ simplifies as

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{s^2 + u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2 + t^2}{u^2} \right)$$

For $m = 0$ the Mandelstam variables are

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1 - \cos \theta) \\ u &= -2E^2(1 + \cos \theta) \end{aligned}$$

Hence

$$\langle |\mathcal{M}|^2 \rangle = 2e^4 \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)} \right)$$

This can be written more compactly as

$$\langle |\mathcal{M}|^2 \rangle = 4e^4 \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{e^4} \frac{\langle |\mathcal{M}|^2 \rangle}{4s} = \frac{\alpha^2}{4E^2} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Multiply by $(\hbar c)^2$ to convert from natural units to physical units.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta} (\hbar c)^2$$

Zee p. 134 has the cross section formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{4\pi} \right)^2 \frac{1}{8E^2} f(\theta)$$

where $f(\theta)$ is the function

$$f(\theta) = \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2) \cos^2(\theta/2)} + \frac{1 + \sin^4(\theta/2)}{\cos^4(\theta/2)}$$

Function $f(\theta)$ is equivalent to

$$f(\theta) = 2 \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Noting that in natural units $e^2 = 4\pi\alpha$ we have

$$\left(\frac{e^2}{4\pi} \right)^2 = \alpha^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{(\cos^2 \theta + 3)^2}{\sin^4 \theta}$$

Eigenmath script