Matrix mechanics 2

Show that for hydrogen the indicated elements of angular momentum matrices are[1]

$$L_1 = \begin{array}{c} \\ \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 & \cdots \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} & \cdots \\ 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

$$L_2 = \begin{array}{c} \\ \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{i\hbar}{\sqrt{2}} & 0 & \cdots \\ 0 & -\frac{i\hbar}{\sqrt{2}} & 0 & \frac{i\hbar}{\sqrt{2}} & \cdots \\ 0 & 0 & -\frac{i\hbar}{\sqrt{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

$$L_3 = \begin{array}{c} \\ \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -\hbar & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \hbar & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

and

$$\mathbf{L}^2 = \begin{array}{c} \\ \psi_{1,0,0} \\ \psi_{2,1,-1} \\ \psi_{2,1,0} \\ \psi_{2,1,1} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \psi_{1,0,0} & \psi_{2,1,-1} & \psi_{2,1,0} & \psi_{2,1,1} & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\hbar^2 & 0 & 0 & \cdots \\ 0 & 0 & 2\hbar^2 & 0 & \cdots \\ 0 & 0 & 0 & 2\hbar^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

Verify

$$\mathbf{L}^2 = {L_1}^2 + {L_2}^2 + {L_3}^2$$

and

$$L_2L_3 - L_3L_2 = i\hbar L_1$$
$$L_3L_1 - L_1L_3 = i\hbar L_2$$
$$L_1L_2 - L_2L_1 = i\hbar L_3$$

[1]See *Quantum Mechanics in Matrix Form* by Günter Ludyk, p. 73.

In spherical coordinates the angular momentum operators are

$$\begin{aligned}
\hat{L}_1 &= i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right) \\
\hat{L}_2 &= i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right) \\
\hat{L}_3 &= -i\hbar \frac{\partial}{\partial\phi}
\end{aligned}$$

and

$$\hat{\mathbf{L}}^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$$