Laplace transform example

Solve for $c_a(t)$ and $c_b(t)$ where $c_a(0) = 1$, $c_b(0) = 0$, and

$$\begin{aligned}\dot{c}_a(t) &= Ae^{-i\omega t}c_b(t)\\ \dot{c}_b(t) &= Be^{i\omega t}c_a(t)\end{aligned}$$

Start with Laplace transforms.

$$\begin{aligned}sC_a(s) - c_a(0) &= AC_b(s + i\omega)\\ sC_b(s) - c_b(0) &= BC_a(s - i\omega)\end{aligned}$$

Solve for $C_a(s)$ with $c_a(0) = 1$.

$$C_a(s) = \frac{AC_b(s + i\omega)}{s} + \frac{1}{s}$$

Solve for $C_a(s - i\omega)$.

$$C_a(s - i\omega) = \frac{AC_b(s)}{s - i\omega} + \frac{1}{s - i\omega}$$

Multiply by B to obtain $sC_b(s)$.

$$sC_b(s) = \frac{ABC_b(s)}{s - i\omega} + \frac{B}{s - i\omega}$$

Rearrange as

$$sC_b(s) - \frac{ABC_b(s)}{s - i\omega} = \frac{B}{s - i\omega}$$

Rearrange again as

$$C_b(s)\left(s - \frac{AB}{s - i\omega}\right) = \frac{B}{s - i\omega}$$

Hence

$$C_b(s) = \frac{\dfrac{B}{s - i\omega}}{s - \dfrac{AB}{s - i\omega}} = \frac{B}{s^2 - i\omega s - AB}$$

Inverse Laplace transform:

$$\frac{1}{s^2 + as + b} \quad \Rightarrow \quad \frac{2}{k}\sin\left(\frac{kt}{2}\right)\exp\left(-\frac{at}{2}\right), \quad k = \sqrt{4b - a^2}$$

Hence for $a = -i\omega$ and $b = -AB$ we have

$$c_b(t) = \frac{2B}{k}\sin\left(\frac{kt}{2}\right)\exp\left(\frac{i\omega t}{2}\right)$$

where

$$k = \sqrt{-4AB + \omega^2}$$

Solve for $c_a(t)$.

$$c_a(t) = \frac{\dot{c}_b(t)}{Be^{i\omega t}} = \left[\cos\left(\frac{kt}{2}\right) + \frac{i\omega}{k}\sin\left(\frac{kt}{2}\right)\right]\exp\left(-\frac{i\omega t}{2}\right)$$