Klein-Gordon equation

Start with relativistic energy.

$$E = \sqrt{p^2c^2 + m^2c^4}$$

Square both sides.

$$E^2 = p^2c^2 + m^2c^4$$

Substitute $i\hbar\partial/\partial t$ for E and $-i\hbar\nabla$ for p.

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 = \left(-i\hbar\nabla\right)^2 c^2 + m^2c^4$$

Reform the operators.

$$-\hbar^2\frac{\partial^2}{\partial t^2} = -\hbar^2c^2\nabla^2 + m^2c^4$$

Rearrange.

$$\hbar^2\left(-\frac{\partial^2}{\partial t^2} + c^2\frac{\partial^2}{\partial x^2} + c^2\frac{\partial^2}{\partial y^2} + c^2\frac{\partial^2}{\partial z^2}\right) = m^2c^4$$

Hence the Klein-Gordon equation.

$$\hbar^2\left(-\frac{\partial^2}{\partial t^2} + c^2\frac{\partial^2}{\partial x^2} + c^2\frac{\partial^2}{\partial y^2} + c^2\frac{\partial^2}{\partial z^2}\right)\psi = m^2c^4\psi$$

Let P be the momentum vector

$$P = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Let X be the position vector

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Let ψ be the scalar function

$$\psi = A\exp\left(-\frac{i}{\hbar}P\cdot X\right) + B\exp\left(\frac{i}{\hbar}P\cdot X\right)$$

where

$$P\cdot X = P^\mu g_{\mu\nu}X^\nu = Et - p_x x - p_y y - p_z z$$

Show that ψ solves the Klein-Gordon equation.

Verify dimensions.

$$\hbar^2 \frac{\partial^2}{\partial t^2} = [\mathrm{J\,s}]^2 \frac{1}{[\mathrm{s}]^2} = [\mathrm{J}]^2$$

$$\hbar^2 c^2 \frac{\partial}{\partial x^2} = [\mathrm{J\,s}]^2 \frac{[\mathrm{m}]^2}{[\mathrm{s}]^2} \frac{1}{[\mathrm{m}]^2} = [\mathrm{J}]^2$$

$$m^2 c^4 = [\mathrm{J}]^2$$

Eigenmath script