Hydrogen transition 1

Start with the perturbing Hamiltonian

$$H_1(\mathbf{r},t) = -\frac{eA_0\boldsymbol{\epsilon}\cdot\mathbf{p}}{m_e}\cos(-\mathbf{k}\cdot\mathbf{r}+\omega t)$$

In exponential form

$$H_1(\mathbf{r},t) = -\frac{eA_0\boldsymbol{\epsilon}\cdot\mathbf{p}}{m_e}\left(\tfrac{1}{2}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)+\tfrac{1}{2}\exp(-i\mathbf{k}\cdot\mathbf{r}+i\omega t)\right)$$

Given the initial condition $c_b(0)=0$ the first-order approximation for $c_b(t)$ is

$$c_b(t) = -\frac{i}{\hbar}\int_0^t \langle\psi_b|H_1|\psi_a\rangle\exp(i\omega_0 t')\,dt',\quad \omega_0=\frac{E_b-E_a}{\hbar}$$

Factor the integrand.

$$\begin{aligned}c_b(t) &= \frac{ieA_0}{2\hbar m_e}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle \\ &\qquad\times\int_0^t\left(\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)+\exp(-i\mathbf{k}\cdot\mathbf{r}+i\omega t)\right)\exp(i\omega_0 t')\,dt'\end{aligned}$$

Solve the integral to obtain

$$\begin{aligned}c_b(t) &= \frac{eA_0}{2\hbar m_e}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega} \\ &\qquad+\frac{eA_0}{2\hbar m_e}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(-i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0+\omega)t\big)-1}{\omega_0+\omega}\end{aligned} \tag{1}$$

As an approximation, discard the second term since the first term dominates for $\omega\approx\omega_0$.

$$c_b(t) = \frac{eA_0}{2\hbar m_e}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\exp\big(i(\omega_0-\omega)t\big)-1}{\omega_0-\omega}$$

Rewrite $c_b(t)$ in the form of a sine function.

$$c_b(t) = \frac{ieA_0}{\hbar m_e}\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\frac{\sin\big(\frac{1}{2}(\omega_0-\omega)t\big)}{\omega_0-\omega}\exp\big(\tfrac{i}{2}(\omega_0-\omega)t\big) \tag{2}$$

Hence the probability to go from state a to b is

$$\Pr_{a\to b}(t) = |c_b(t)|^2 = \frac{e^2|A_0|^2}{\hbar^2 m_e^2}\big|\langle\psi_b|\boldsymbol{\epsilon}\cdot\mathbf{p}\exp(i\mathbf{k}\cdot\mathbf{r})|\psi_a\rangle\big|^2\frac{\sin^2\big(\frac{1}{2}(\omega_0-\omega)t\big)}{(\omega_0-\omega)^2}$$