Hydrogen atom statistics 1

Consider the ground state wavefunction

$$\psi(r) = \frac{1}{\sqrt{\pi a_0^3}} \exp\left(-\frac{r}{a_0}\right)$$

The cumulative distribution function is

$$\Pr(r \le a) = \int_0^a \int_\Omega |\psi(r)|^2 \, dr \, d\Omega = \frac{1}{\pi a_0^3} \int_0^a \int_0^\pi \int_0^{2\pi} \exp\left(-\frac{2r}{a_0}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Integrate over θ and ϕ (multiply by 4π).

$$\Pr(r \le a) = \frac{4}{a_0^3} \int_0^a \exp\left(-\frac{2r}{a_0}\right) r^2 \, dr$$

Integrate over r.

$$\Pr(r \le a) = \frac{4}{a_0^3} \exp\left(-\frac{2r}{a_0}\right) \left(-\frac{1}{2} a_0 r^2 - \frac{1}{2} a_0^2 r - \frac{1}{4} a_0^3\right) \Bigg|_{r=0}^{r=a}$$

Hence

$$\Pr(r \le a) = 1 - \left(\frac{2a^2}{a_0^2} + \frac{2a}{a_0} + 1\right) \exp\left(-\frac{2a}{a_0}\right)$$

For $a = a_0$ we have

$$\Pr(r \le a_0) = 0.32$$

Hence the probability that the electron is inside the Bohr radius is 32%.

Eigenmath script