How Planck calculated h and k

Max Planck used two experimental results to calculate h and k in his 1901 paper "On the Law of Distribution of Energy in the Normal Spectrum." Although the quantum of action h is well known as Planck's constant, the use of k for Boltzmann's constant is also due to Planck. In addition, Planck was the first to compute a numerical value for k.

One of the experimental results Planck used was the difference $S_{100} - S_0$ determined by Ferdinand Kurlbaum in 1898 where S_t is the power radiated by a black body at t degrees Celsius.

$$S_{100} - S_0 = 7.31 \times 10^5 \,\mathrm{erg}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$$

From the radiant power formula $S_t = (t + 273)^4\sigma$ we have

$$S_{100} - S_0 = (100 + 273)^4\sigma - (0 + 273)^4\sigma = (373^4 - 273^4)\sigma$$

Hence the Stefan-Boltzmann constant σ can be determined from $S_{100} - S_0$.

$$\sigma = \frac{S_{100} - S_0}{373^4 - 273^4}$$

The Stefan-Boltzmann law is the relation between energy density and temperature θ.

$$\text{“energy per unit volume”} = \frac{4\sigma\theta^4}{c}$$

The use of θ for temperature looks strange but that is what scientists used at the time.

Using the Stefan-Boltzmann law and Kurlbaum's measurement, Planck calculated energy density for temperature $\theta = 1$.

$$\frac{4}{c} \times \frac{S_{100} - S_0}{373^4 - 273^4} = \frac{4}{3 \times 10^{10}} \times \frac{7.31 \times 10^5}{373^4 - 273^4} = 7.061 \times 10^{-15} \,\mathrm{erg}\,\mathrm{cm}^{-3}$$

Planck's 1901 paper has the following formula (Equation 12) for energy distribution u as a function of frequency ν and temperature θ.

$$u = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k\theta} - 1}$$

The integral of u over all frequencies yields the total energy density u^*.

$$u^* = \int_0^\infty u \, d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/k\theta} - 1} \, d\nu$$

Planck used a series expansion to solve the integral for $\theta = 1$. However, we will use the following identity.

$$\int_0^\infty \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15}$$

By the change of variable $x = h\nu/k$ we have

$$u^* = \frac{8\pi h}{c^3}\left(\frac{k}{h}\right)^4 \frac{\pi^4}{15}$$

Planck then set u^* equal to the result from the Stefan-Bolztmann law.

$$\frac{8\pi h}{c^3}\left(\frac{k}{h}\right)^4 \frac{\pi^4}{15} = 7.061 \times 10^{-15}$$

Hence

$$\frac{k^4}{h^3} = 7.061 \times 10^{-15} \times \frac{15c^3}{8\pi^5} = 1.1682 \times 10^{15}$$

The second experimental result Planck used was $\lambda_m \theta = 0.294$ obtained in 1900 by Otto Lummer and Ernst Pringsheim. Symbol λ_m is the wavelength in centimeters of peak radiant energy for a black body at temperature θ in Kelvin.

Planck's 1901 paper has the following formula (Equation 13) for energy distribution E as a function of wavelength λ and temperature θ.

$$E = \frac{8\pi ch}{\lambda^5} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Planck solves $dE/d\lambda = 0$ to obtain λ_m which we will now do step by step. First, compute $dE/d\lambda$.

$$\frac{dE}{d\lambda} = \frac{8\pi c^2 h^2}{k\lambda^7 \theta} \frac{e^{ch/k\lambda\theta}}{(e^{ch/k\lambda\theta} - 1)^2} - \frac{40\pi ch}{\lambda^6} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Set $dE/d\lambda = 0$ to obtain

$$\frac{8\pi c^2 h^2}{k\lambda^7 \theta} \frac{e^{ch/k\lambda\theta}}{(e^{ch/k\lambda\theta} - 1)^2} = \frac{40\pi ch}{\lambda^6} \frac{1}{e^{ch/k\lambda\theta} - 1}$$

Then by cancellation of terms

$$\frac{ch}{5k\lambda\theta} \frac{e^{ch/k\lambda\theta}}{e^{ch/k\lambda\theta} - 1} = 1$$

Multiply both sides by $e^{ch/k\lambda\theta} - 1$.

$$\frac{ch}{5k\lambda\theta} e^{ch/k\lambda\theta} = e^{ch/k\lambda\theta} - 1$$

Subtract $e^{ch/k\lambda\theta}$ from both sides.

$$\left(\frac{ch}{5k\lambda\theta} - 1\right) e^{ch/k\lambda\theta} = -1$$

Multiply both sides by -1 to obtain Planck's result.

$$\left(1 - \frac{ch}{5k\lambda\theta}\right) e^{ch/k\lambda\theta} = 1$$

Planck then provides the following numerical solution.

$$\frac{ch}{k\lambda\theta} = 4.9651$$

Then using $c = 3 \times 10^{10}$ and $\lambda\theta = 0.294$ Planck calculates

$$\frac{h}{k} = 4.9651 \times \frac{\lambda\theta}{c} = 4.9651 \times \frac{0.294}{3 \times 10^{10}} = 4.866 \times 10^{-11}$$

Planck then solves for k. Plug $h/k = 4.866 \times 10^{-11}$ into the formula for k^4/h^3 to obtain

$$k = 1.1682 \times 10^{15} \times \frac{h^3}{k^3} = 1.1682 \times 10^{15} \times (4.866 \times 10^{-11})^3 = 1.346 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$$

Then calculate h directly from k.

$$h = k \times 4.866 \times 10^{-11} = 6.55 \times 10^{-27}\,\mathrm{erg\,s}$$